# 18.901– Introduction to topology

## Midterm 1

## MIT

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3/6/25

**Name**: Solutions

**Student Number**: ________________

- This exam contains 20 pages and 13 questions.
- This exam is out of 68 points. The distribution of points among all of the questions is shown in the table on the page 2 and is also indicated next to each question.
- Do NOT write on the backs of any pages. There are additional pages at the end of the exam if you should need them to show further work. Please indicate in your solutions when we should refer to the pages at the end of the exam for more details.
- You will have 80 minutes to complete the exam.
- Good luck!

**Distribution of Marks**

| Question | Points | Score |
|---|---|---|
| 1 | 3 | |
| 2 | 3 | |
| 3 | 4 | |
| 4 | 3 | |
| 5 | 3 | |
| 6 | 4 | |
| 7 | 4 | |
| 8 | 4 | |
| 9 | 4 | |
| 10 | 10 | |
| 11 | 8 | |
| 12 | 8 | |
| 13 | 10 | |
| Total: | 68 | |

# 1 Definitions

1. (3 points) Give the definition of a basis on a set $X$.

A basis on a set $X$ is a set of subsets of $X$, say $\mathcal{B}$, st

① (covering) $X = \bigcup_{B \in \mathcal{B}} B$.

② (refinement) If $x \in B' \cap B''$ for $B', B'' \in \mathcal{B}$, then $\exists\, B \in \mathcal{B}$ st $x \in B \subseteq B' \cap B''$.

2. (3 points) State three different (but equivalent) formulations of when a $T_1$ space is normal.

Let $X$ be a $T_1$ space.

① For all closed subsets $A \subseteq X$, $B \subseteq X$ st $A \cap B = \emptyset$ there exist open subsets $U \subseteq X$, $V \subseteq X$ st $A \subseteq U$, $B \subseteq V$, and $U \cap V = \emptyset$.

② For all closed subsets $A \subseteq X$ and open subsets $U \subseteq X$ st $A \subseteq U$ there exists an open subset $V \subseteq X$ st $A \subseteq V \subseteq \overline{V} \subseteq U$.

③ For all closed subsets $A \subseteq X$, $B \subseteq X$ st $A \cap B = \emptyset$ there exists a continuous map $f : X \to I$ st $f|_A \equiv 0$ and $f|_B \equiv 1$.

3. (4 points) State four different (but equivalent) formulations of when a map $f\colon X \to Y$ of spaces is continuous.

① For all open subsets $\mathcal{U} \subseteq Y$, $f^{-1}(\mathcal{U}) \subseteq X$ is open.

② For all closed subsets $C \subseteq Y$, $f^{-1}(C) \subseteq X$ is closed.

③ For all open subsets $V \subseteq Y$, if $f(x) \in V$ for some $x \in X$, then there exists an open subset $\mathcal{U} \subseteq X$ st $x \in \mathcal{U}$ and $f(\mathcal{U}) \subseteq V$.

④ For all subsets $A \subseteq X$, $f(\overline{A}) \subseteq \overline{f(A)}$.

## 2 True/False

For Questions 4-9, state whether or not the given statement is true or false. If it is true, provide a proof of the statement. If it is false, provide a counter-example to the statement.

4. (3 points) Let $X$ be a space. Let $A \subseteq X$ be an open subspace. Let $B \subseteq A$ be a subspace. The interior of $B$ in $A$ is given by the intersection of $A$ with the interior of $B$ in $X$.

True:

$$\mathrm{int}_A(B) = \bigcup_{\substack{V = A\cap U \subseteq B \\ U \subseteq X \text{ open}}} V \overset{(1)}{=} \bigcup_{\substack{V \subseteq B \\ V \subseteq X \text{ open}}} V \overset{(2)}{=} A \cap \bigcup_{\substack{V \subseteq B \\ V \subseteq X \text{ open}}} V = A \cap \mathrm{int}_X(B)$$

(1) Since $A$ is open and $B \subseteq A$, every open in $X$ that is contained in $B$ is of the form $A \cap U$ for $U \subseteq X$ an open subset w/ $A \cap U \subseteq B$.

(2) Since LHS is in $A$, we have this equality.

5. (3 points) Let $\mathcal{O}$ and $\mathcal{O}'$ be two topologies on a space $X$ such that $\mathcal{O}$ is coarser than $\mathcal{O}'$. If $(X, \mathcal{O})$ is compact, then $(X, \mathcal{O}')$ is compact.

False :

Let $X = \mathbb{R}$, $\mathcal{O}$ = trivial topology, $\mathcal{O}'$ = standard topology.

- The trivial topology, being $\{\phi, X\}$, is always coarser than any other topology. So $\mathcal{O} \subseteq \mathcal{O}'$.
- $(X, \mathcal{O})$ is compact because the only open cover is $X$, which is finite.
- $(X, \mathcal{O}')$ is non-compact via considering the covering $X = \bigcup_n (-n, n)$.

6. (4 points) If $f\colon X \to Y$ is a continuous map of spaces, then

$$\partial f^{-1}(B) \subseteq f^{-1}(\partial B)$$

for every subset $B \subseteq Y$.

<u>True</u>:

Notice that $\forall A \subseteq X$, $\partial A = \overline{A} \cap \overline{X \setminus A}$.

Since $f$ is continuous,

$$f(\overline{f^{-1}(B)}) = \overline{f(f^{-1}(B))} \subseteq \overline{B}$$

$$f(\overline{X \setminus f^{-1}(B)}) = \overline{f(X \setminus f^{-1}(B))} = \overline{f(f^{-1}(Y \setminus B))} \subseteq \overline{Y \setminus B}$$

$\Rightarrow$ $f(\partial f^{-1}(B)) \subseteq \partial B$

$\Rightarrow$ $\partial f^{-1}(B) \subseteq f^{-1}(f(\partial f^{-1}(B))) \subseteq f^{-1}(\partial B)$

7. (4 points) Let $X$ be a space. If $A \subseteq X$ is connected, then $\overline{A}$ is connected.

<u>True</u>:

Spse $\overline{A} = C_0 \sqcup C_1$ w/ $C_i$ closed in $\overline{A}$.

Since $\overline{A} \subseteq X$ is closed, $C_i$ is closed in $X$.

$\Rightarrow$ $C_i' = A \cap C_i$ is closed in $A$.

Since $A = C_0' \sqcup C_1'$ and $A$ is connected, $C_i' = A$ for some $i$.

$\Rightarrow$ $A \subseteq C_i$ = closed $\Rightarrow$ $\overline{A} \subseteq C_i$ $\Rightarrow$ $C_{j \neq i} = \emptyset$

$\Rightarrow$ $\overline{A}$ is connected.

8. (4 points) Let $X$ be a connected Hausdorff space. Let $f\colon X \to X$ be a continuous map and set

$$\operatorname{fix}(f) = \{x \in X \mid f(x) = x\}.$$

For every non-empty open subset $U \subseteq X$, there exists a unique continuous map $f\colon X \to X$ with $\operatorname{fix}(f) = U$.

False

Spse $f : X \to X$ is a continuous map w/ $\operatorname{fix}(f) = \mathcal{U}$.

Since $X$ is Hausdorff, $\Delta \leq X \times X$ is closed.

Define $F : X \to X \times X$ by $F(x) = (x, f(x))$.

Since $F$ is component-wise continuous, $F$ is continuous.

Note, open + non-empty $= \mathcal{U} = \operatorname{fix}(f) = F^{-1}(\Delta) =$ closed.

Since $X$ is connected, $X = \operatorname{fix}(f) = \mathcal{U}$.

$\Rightarrow \exists$ such a $f$ iff $\mathcal{U} = X$.

So $X = \mathbb{R}$ and $\mathcal{U} = (0,1)$ gives a counter-example.

9. (4 points) Let $X$ be a connected compact Hausdorff space with $|X| > 1$. $X$ is uncountable.

True :

Fix $x, y \in X$ st $x \neq y$.

Since $X$ is normal, it is $T_1$. So $\{x\}$, $\{y\}$ are closed.

By Urysohn's Lemma, $\exists$ continuous map $f: X \to I$ st $f(x) = 0$, $f(y) = 1$.

Since $X$ is connected, $f(X)$ is connected.

If $\exists\, z \in I \setminus f(X)$, then $f(X) = (f(X) \cap [0, z)) \sqcup (f(X) \cap (z, 1])$, which would contradict that $f(X)$ is connected.

So $f$ is surjective $\Rightarrow |X| \geq |\mathbb{R}| \Rightarrow X$ is uncountable.

# 3 Free response

10. Consider $X = \mathbb{R}^n$ with the standard topology, denoted $\mathcal{O}_{std}$. Let $\mathcal{O}^*$ denote a topology on $X \sqcup \{\infty\}$ where $U \in \mathcal{O}^*$ if and only if either $U \in \mathcal{O}_{std}$ or $U = (X \setminus K) \sqcup \{\infty\}$ where $K \subseteq \mathbb{R}^n$ is a compact subspace.

    (a) (7 points) Show that $\mathcal{O}^*$ defines a topology.

(1) Note $K = \emptyset$ is compact. So $X \cup \{\infty\} \in \mathcal{O}^*$.

Note $\emptyset \in \mathcal{O}_{std}$. So $\emptyset \in \mathcal{O}^*$.

(2) Let $U_\alpha = (X \setminus C_\alpha) \in \mathcal{O}_{std}$, $V_\beta = (X \setminus K_\beta) \cup \{\infty\}$ w/ $K_\beta = \text{cpt}$.

If $\{\beta\} = \emptyset$, then $\bigcup_\alpha U_\alpha \in \mathcal{O}_{std} \subseteq \mathcal{O}^*$

Else, $\bigcup_\alpha U_\alpha \cup \bigcup_\beta V_\beta = \{\infty\} \sqcup \left( X \setminus \left( \bigcap_\alpha C_\alpha \cap \bigcap_\beta K_\beta \right) \right)$

Note, by Heine-Borel, each $K_\beta = \text{closed} \Rightarrow \bigcap_\alpha C_\alpha \cap \bigcap_\beta K_\beta = \text{closed}$.

By Heine-Borel, each $K_\beta = \text{bounded} \Rightarrow \bigcap_\beta K_\beta = \text{bounded}$

$\Rightarrow \bigcap_\alpha C_\alpha \cap \bigcap_\beta K_\beta = \text{bounded}$.

So by Heine-Borel, $\bigcap_\alpha C_\alpha \cap \bigcap_\beta K_\beta$ is compact.

$\Rightarrow \bigcup_\alpha U_\alpha \cup \bigcup_\beta V_\beta \in \mathcal{O}^*$

(3) Let $U_i = X \setminus C_i \in \mathcal{O}_{std}$, $V_j = (X \setminus K_j) \cup \{\infty\}$ w/ $K_j = \text{cpt}$.

If $\{i\} = \emptyset$, then $\bigcap_j V_j = \{\infty\} \sqcup (X \setminus \bigcup_j K_j)$.

Since there is a finite # of $K_j$ and since each $K_j$ is closed + bounded by Heine-Borel, $\bigcup_j K_j$ is closed + bounded and, thus, compact by Heine-Borel. So $\bigcap_j V_j \in \mathcal{O}^*$.

If $\{i\} \neq \emptyset$, $\bigcap_i U_i \cap \bigcap_j V_j = (X \setminus \bigcup_i C_i \cup \bigcup_j K_j)$.

Since there is a finite # of $K_j, C_i$ and since each $K_j$ is closed by Heine-Borel, $\bigcup_i C_i \cup \bigcup_j K_j$ is closed in $\mathcal{O}_{std}$.

$\Rightarrow \bigcap_i U_i \cap \bigcap_j V_j \in \mathcal{O}_{std} \subseteq \mathcal{O}^*$.

$(\mathbb{R}^n \sqcup \{\infty\}, \mathcal{O}^*)$

(b) (3 points) Show that $\cancel{(\mathbb{R}^n, \mathcal{O}^*)}$ is compact.

Let $U_\alpha \in \mathcal{O}_{std}$, $V_\beta := (X \setminus K_\beta) \sqcup \{\infty\}$ be an open cover.

Since they cover, they must cover $\{\infty\}$. So $\exists\, \gamma \in \{\beta\}$.

Since $K_\beta \subseteq \mathbb{R}^n$ is cpt, it is closed.

So $U_\alpha$, $X \setminus K_\beta$ give an open cover of $\mathbb{R}^n$.

Since $K_\gamma \subseteq \mathbb{R}^n$ is cpt, $\exists\, U_1, \ldots, U_n$ and $X \setminus K_1, \ldots, X \setminus K_m$ st

$$K_\gamma \subseteq \bigcup_i U_i \cup \bigcup_j X \setminus K_j$$

So $U_1, \ldots, U_n$, $V_1, \ldots, V_m$, $V_\gamma$ cover $X$.

11. Let $X$ and $Y$ be compact Hausdorff spaces.

(a) (5 points) Show that $X \times Y$ is compact.

Spse $X \times Y = \bigcup_\alpha O_\alpha$ is an open cover.

Write $O_\alpha = \bigcup_{\alpha_\beta} U_{\alpha\beta} \times V_{\alpha\beta}$.

Note if the cover $X = \bigcup_{\alpha,\beta} U_{\alpha\beta} \times V_{\alpha\beta}$ admits a finite subcover, say by $U_{i_j} \times V_{i_j}$,

then $O_i$ is a finite subcover of the $O_\alpha$.

Since $\{x\} \times Y$ is compact, $\exists$ finite # $V_{x,i} \subseteq \{V_{\alpha\beta}\}$ st $U_{x,i} \times V_{x,i}$ cover $\{x\} \times Y$.

Set $U_x = \bigcap_i U_{x,i}$.

Since $X$ is compact, $\exists$ finite # $U_{x_j} \subseteq \{U_x\}$ that cover $X$.

So $U_{x_j,i} \times V_{x_j,i}$ cover $X \times Y$ as a finite subcover. Indeed, given $(x,y) \in X \times Y$,

$x \in U_{x_j} \subseteq U_{x_j,i}$ for some $j$ and $y \in V_{x_j,i}$ for some $i$. So $(x,y) \in U_{x_j,i} \times V_{x_j,i}$.

(b) (3 points) Show that $X \times Y$ is Hausdorff.

Fix unique points $(x_0, y_0), (x_1, y_1) \in X \times Y$.

WLOG, assume $x_0 \neq x_1$.

Since $X$ is Hausdorff, $\exists$ disjoint opens $U_0, U_1 \subseteq X$ st $x_i \in U_i$.

Note $U_0 \times Y$, $U_1 \times Y$ are disjoint opens in $X \times Y$ st $(x_i, y_i) \in U_i \times Y$.

$\Rightarrow$ $X \times Y$ is Hausdorff.

12. (8 points) Let $X$ be a compact Hausdorff space with basis $\mathcal{B}$. Show that for all $x \in X$ and opens $U \subseteq X$ such that $x \in U$ there exists a basis element $B \in \mathcal{B}$ such that $x \in B \subseteq \overline{B} \subseteq U$. (**You may not use any results from class.**)

Set $A = X \setminus U$ = closed.

Since $X$ = Haus, $\forall y \in A$, $\exists$ disjoint opens $W_y, V_y$ st $x \in W_y$, $y \in V_y$.

Since $A$ is closed and $X$ is compact, $A$ is compact.

Note, $A \subseteq \bigcup_y V_y$. So there exists a finite subcover $V_{y_1}, \dots, V_{y_n}$.

Define $W = \bigcap_i W_{y_i}$, $V = \bigcup_i V_{y_i}$.

Note, $x \in W_y \; \forall y \Rightarrow x \in W$.

Since $V_{y_1}, \dots, V_{y_n}$ are a finite subcover, $A \subseteq V$.

If $z \in V$, then $z \in V_{y_i}$ for some $i \Rightarrow z \notin W_{y_i}$ for some $i \Rightarrow z \notin \bigcap_i W_{y_i}$.

$\Rightarrow W \cap V = \phi$.

Note, $x \subseteq W \subseteq X \setminus V$ = closed. So $x \subseteq W \subseteq \overline{W} \subseteq X \setminus V$.

Since $V \supseteq A \Rightarrow X \setminus V \subseteq X \setminus A = U$. So $x \subseteq W \subseteq \overline{W} \subseteq U$.

$W$ = open $\Rightarrow \exists$ basis elm $B \in \mathcal{B}$ st $x \in B \subseteq W$.

$\Rightarrow \overline{B} \subseteq \overline{W}$ since $\overline{B}$ = smallest closed subset that contains $B$.

$\Rightarrow x \in B \subseteq \overline{B} \subseteq U$.

13. (10 points) Let
$$\mathbb{D}^2 = \{x \in \mathbb{R}^2 \mid |x| \leq 1\}.$$
Suppose that we have spaces $X_i$ for $i = 1, 2$ and homeomorphisms $\varphi_i \colon X_i \to \mathbb{D}^2$.

Let $X = X_1 \sqcup X_2$ denote the topological disjoint union. Define an equivalene relation $\sim$ on $X$ by $x \sim y$ if and only if $x = y$ or $x \in X_i$, $y \in X_j$, $\varphi_i(x) = \varphi_j(y)$, and $|\varphi_i(x)| = 1$. Show that $X/\sim$ is homeomorphic to $S^2$. (Hint: You may assume that $\sqrt{\cdot} \colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is continuous.)

- Define $f_1 : X_1 \to S^2$ by $f_1(x) = \left(\varphi_1(x), \sqrt{1 - |\varphi_1(x)|^2}\right)$

  Note, $|\cdot|^2 : \mathbb{R}^2 \to \mathbb{R}$, $|(x,y)|^2 = x^2 + y^2$ is continuous since it is a sum/product/composition of continuous functions.

  So $\sqrt{1 - |\varphi_1(\cdot)|^2}$ is continuous since it is a sum/product/composition of continuous functions.

  It follows that $f_1 : X \to \mathbb{R}^2 \times \mathbb{R}$ is component-wise continuous.

  Since $f(X) \subseteq S^2$, the restriction to $S^2$ is continuous.
- Similarly define $f_2 : X_2 \to S^2$ by $\left(\varphi_2(x), -\sqrt{1 - |\varphi_2(x)|^2}\right)$.
- By Pasting Lemma (or just defn of topology of $X$),
$$f : X \to S^2 \qquad f(x) = \begin{cases} f_1(x) & , x \in X_1 \\ f_2(x) & , x \in X_2 \end{cases}$$
  is continuous.
- $f_i$ has inverse given by $(x, y, z) \mapsto \varphi_i^{-1}(x, y)$.

  So $f_i$ is bijective.
- Note if $x \sim y$, then $f(x) = f(y)$.

  $\Rightarrow \exists\ \bar{f} : X/\sim \to S^2$ st $\bar{f} \circ q = f$.

  $\bar{f}$ is cts since $f$ is cts + $\bar{f}$ is surjective since $q, f$ are surjective.
- If $\bar{f}([x]) = \bar{f}([y])$ w/ $x \neq y$, then $\varphi_i(x) = \varphi_j(y)$ and $|\varphi_i(x)| = \underline{1}$.

  $\Rightarrow x \sim y$. So $\bar{f}$ is injective.
- $$\left.\begin{array}{l} X_1 \sqcup X_2 = \text{cpt} \Rightarrow X_1 \sqcup X_2/\sim = \text{cpt} \\ S^2 \subseteq \mathbb{R}^3 = \text{Haus} \Rightarrow S^2 = \text{Haus} \end{array}\right\} \Rightarrow \bar{f} = \text{homeo}.$$

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