# 18.901– Introduction to topology

# Midterm 1

# MIT

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3/6/25

**Name**: ______________________________

**Student Number**: ______________________________

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- This exam contains 20 pages and 13 questions.
- This exam is out of 68 points. The distribution of points among all of the questions is shown in the table on the page 2 and is also indicated next to each question.
- Do NOT write on the backs of any pages. There are additional pages at the end of the exam if you should need them to show further work. Please indicate in your solutions when we should refer to the pages at the end of the exam for more details.
- You will have 80 minutes to complete the exam.
- Good luck!

**Distribution of Marks**

| Question | Points | Score |
|---|---|---|
| 1 | 3 | |
| 2 | 3 | |
| 3 | 4 | |
| 4 | 3 | |
| 5 | 3 | |
| 6 | 4 | |
| 7 | 4 | |
| 8 | 4 | |
| 9 | 4 | |
| 10 | 10 | |
| 11 | 8 | |
| 12 | 8 | |
| 13 | 10 | |
| Total: | 68 | |

# 1 Definitions

1. (3 points) Give the definition of a basis on a set $X$.

2. (3 points) State three different (but equivalent) formulations of when a $T_1$ space is normal.

3. (4 points) State four different (but equivalent) formulations of when a map $f\colon X \to Y$ of spaces is continuous.

# 2 True/False

For Questions 4-9, state whether or not the given statement is true or false. If it is true, provide a proof of the statement. If it is false, provide a counter-example to the statement.

4. (3 points) Let $X$ be a space. Let $A \subseteq X$ be an open subspace. Let $B \subseteq A$ be a subspace. The interior of $B$ in $A$ is given by the intersection of $A$ with the interior of $B$ in $X$.

5. (3 points) Let $\mathcal{O}$ and $\mathcal{O}'$ be two topologies on a space $X$ such that $\mathcal{O}$ is coarser than $\mathcal{O}'$. If $(X, \mathcal{O})$ is compact, then $(X, \mathcal{O}')$ is compact.

6. (4 points) If $f\colon X \to Y$ is a continuous map of spaces, then

$$\partial f^{-1}(B) \subseteq f^{-1}(\partial B)$$

for every subset $B \subseteq Y$.

7. (4 points) Let $X$ be a space. If $A \subseteq X$ is connected, then $\overline{A}$ is connected.

8. (4 points) Let $X$ be a connected Hausdorff space. Let $f\colon X \to X$ be a continuous map and set

$$\mathrm{fix}(f) = \{x \in X \mid f(x) = x\}.$$

For every non-empty open subset $U \subseteq X$, there exists a unique continuous map $f\colon X \to X$ with $\mathrm{fix}(f) = U$.

9. (4 points) Let $X$ be a connected compact Hausdorff space with $|X| > 1$. $X$ is uncountable.

# 3 Free response

10. Consider $X = \mathbb{R}^n$ with the standard topology, denoted $\mathcal{O}_{std}$. Let $\mathcal{O}^*$ denote a topology on $X \sqcup \{\infty\}$ where $U \in \mathcal{O}^*$ if and only if either $U \in \mathcal{O}_{std}$ or $U = (X \setminus K) \sqcup \{\infty\}$ where $K \subseteq \mathbb{R}^n$ is a compact subspace.

    (a) (7 points) Show that $\mathcal{O}^*$ defines a topology.

(b) (3 points) Show that $(\mathbb{R}^n, \mathcal{O}^*)$ is compact.

11. Let $X$ and $Y$ be compact Hausdorff spaces.

    (a) (5 points) Show that $X \times Y$ is compact.

(b) (3 points) Show that $X \times Y$ is Hausdorff.

12. (8 points) Let $X$ be a compact Hausdorff space with basis $\mathcal{B}$. Show that for all $x \in X$ and opens $U \subseteq X$ such that $x \in U$ there exists a basis element $B \in \mathcal{B}$ such that $x \in B \subseteq \overline{B} \subseteq U$. (**You may not use any results from class.**)

13. (10 points) Let

$$\mathbb{D}^2 = \{x \in \mathbb{R}^2 \mid |x| \leq 1\}.$$

Suppose that we have spaces $X_i$ for $i = 1, 2$ and homeomorphisms $\varphi_i \colon X_i \to \mathbb{D}^2$.

Let $X = X_1 \sqcup X_2$ denote the topological disjoint union. Define an equivalene relation $\sim$ on $X$ by $x \sim y$ if and only if $x = y$ or $x \in X_i$, $y \in X_j$, $\varphi_i(x) = \varphi_j(y)$, and $|\varphi_i(x)| = 1$. Show that $X$ is homeomorphic to $S^2$. (Hint: You may assume that $\sqrt{\cdot} \colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is continuous.)

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