Topology - Spring 2025

Alex Pieloch

Contents

Chapter 1

Point-set topology

1.1 Topological spaces and bases

We begin with the definition of a topological space.

Definition 1.1.1. A *topology* on a set X is a set of subsets of X, $\mathcal{O}$, call *open sets* that satisfies:

(1) $\varnothing, X \in \mathcal{O}$.

(2) If $\mathcal{O}' \subseteq \mathcal{O}$, then $\cup_{U \in \mathcal{O}'} U \in \mathcal{O}$.

(3) If $U_1, \ldots, U_n \in \mathcal{O}$, then $\cap_{i=1}^{n} U_i \in \mathcal{O}$.

The pair $(X, \mathcal{O})$ is called a *(topological) space*.

The second condition in Definition 1.1.1 says that arbitrary unions of open sets in X is an open set in X. The third condition in Definition 1.1.1 says that the finite (not infinite!) intersection of open sets in X is an open set in X. Often times we will simple call the set X a (topological) space, where the topology $\mathcal{O}$ is understood.

Example 1.1.2. Consider the set $X = \{1, 2, 3\}$ and the following sets of subsets:

(1) $\{\varnothing, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$. This is not a topology on the set X. The intersection axiom is not satisfied.

(2) $\{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$. This is a topology on the set X. This is the discrete topology on the set X.

(3) $\{\varnothing, \{1\}, \{2, 3\}, \{1, 2, 3\}\}$. This is a topology on the set X.

(4) $\{\varnothing, \{1, 2, 3\}\}$. This is a topology on the set X. This is the trivial topology on the set X.

(5) $\{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2, 3\}\}$. This is not a topology on the set X. The union axiom is not satisfied.

(6) $\{\varnothing, \{2\}, \{3\}, \{2, 3\}, \{1, 2, 3\}\}$. This is a topology on the set X.

(7) $\{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$. This is not a topology on the set X. It does not contain $\{1, 2, 3\}$.

We have a notion of when one topology is courser/finer than another topology on the same set.

Definition 1.1.3. Let $\mathcal{O}$ and $\mathcal{O}'$ be two topologies on a set X. If $\mathcal{O} \subseteq \mathcal{O}'$, then $\mathcal{O}'$ is *finer* than $\mathcal{O}$ and $\mathcal{O}$ is *coarser* than $\mathcal{O}'$.

If $\mathcal{O}'$ is finer than $\mathcal{O}$, then the statement that $\mathcal{O} \subseteq \mathcal{O}'$ says that $\mathcal{O}'$ has more open subsets of X than $\mathcal{O}$. In other words, it's open subsets are *finer*. The analogous explanation explains the term *coarser*.

Example 1.1.4. Every set always has at least two topologies, namely the discrete topology and the trivial topology.

(1) The *discrete topology* on a set X has $\mathcal{O}$ given by the power set of X (the set of all subsets of X).

(2) The *trivial topology* on a set X has $\mathcal{O} = \{\varnothing, X\}$.

The discrete topology on a set is always finer than any other topology on the set. Similarly, the trivial topology is always coarser than any other topology on the set.

Often times, it is very difficult to explicitly express every open subset in a topology. A basis gives us a way of defining a topology using only the basic open subsets (i.e. not requiring us to explicitly specify every open subset). This is defined here.

Definition 1.1.5. A set of subsets $\mathcal{B}$ of X is a *basis* if

(1) $X = \cup_{B \in \mathcal{B}} B$.

(2) If $x \in B' \cap B''$ with $B', B'' \in \mathcal{B}$, then there exists $B \in \mathcal{B}$ such that $x \in B \subseteq B' \cap B''$.

The elements of $\mathcal{B}$ are called *basic opens*.

The first condition in Definition 1.1.5 states that the collection of subsets in a basis covers the set X. The second condition in Definition 1.1.5 is a refinement property: given a point in the intersection of two basic opens, there exists a basic open that is contain in their intersection while also containing the given point.

Lemma 1.1.6. *A basis $\mathcal{B}$ for a set X* generates *a topology $\mathcal{O}$ on X via $U \in \mathcal{O}$ if and only if for all $x \in U$ there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$.*

Proof. We need to verify the axioms of a topology. We verify each axiom in turn.

(1) Since there are no points in $\varnothing$, we vacuously have that $\varnothing \in \mathcal{O}$. To see that $X \in \mathcal{O}$, recall that the subsets in $\mathcal{B}$ cover X. So for each $x \in X$, there exists $B \in \mathcal{B}$ such that $x \in B \subseteq X$. Consequently, $X \in \mathcal{O}$.

(2) Suppose that $U_\alpha \subseteq X$ are open subsets and fix $x \in \cup_\alpha U_\alpha$. Note that $x \in U_\alpha$ for some α. Since U_α is open, we have that there exists $B_\alpha \in \mathcal{B}$ such that $x \in B_\alpha \subseteq U_\alpha \subseteq \cup_\alpha U_\alpha$. Consequently, the union is contained in $\mathcal{O}$.

(3) By induction, it suffices to prove the axiom for the intersection of two open subsets. So suppose that $x \in U_1 \cap U_2$. There exist $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \subseteq U_1$ and $x \in B_2 \subseteq U_2$. By the second condition in Definition 1.1.5, we have that there exists $B \in \mathcal{B}$ such that $x \in B \subseteq B_1 \cap B_2 \subseteq U_1 \cap U_2$. Consequently, we have shown that the intersection is open.

□

Example 1.1.7. Let $X = \mathbb{R}$, the set of real numbers.

(1) The *standard topology* on $\mathbb{R}$ is the topology generated by the basis of open intervals (a, b).

(2) A different basis that generates a different topology on $\mathbb{R}$ is given by the basis of half-open intervals $[a, b)$.

(3) Yet another basis that generates a different topology on $\mathbb{R}$ is given by subsets of the form

$$\{U \subseteq \mathbb{R} \mid U = \mathbb{R} \smallsetminus \{x_1, \dots, x_n\} \text{ for some } x_i \in \mathbb{R}\}.$$

The reader should check for themselves that these sets of subsets each define a basis. The reader can also note the topology generated by the first basis is finer than the topology generated by the third basis. Also the topology generated by the second basis is finer than the topology generated by the first basis.

Remark 1.1.8. We noted in Definition 1.1.1 that we did not require arbitrary intersections of open subsets be open. It is natural to ask whether the definition of a topology actually forces arbitrary intersections of open subsets to open; however, this is not the case. Indeed, consider $\mathbb{R}$ with the standard topology and consider the open subsets $U_n = (-1/n, 1/n)$. Note that $\cap_n U_n = \{0\}$, which is not an open subset since it does not contain any open intervals.

Remark 1.1.9. As hinted at above, often times different bases generate different topologies; however, this is not always the case. It could be that the same topology is generated by multiple different bases. To see this, consider $X = \mathbb{R}^2$ and the two bases given by

(1) $\{B_x(r)\}$, that is, the set of open balls of radii r centered at points x.

(2) $\{Sq_x(r)\}$, that is, the set of open squares with diagonals of length $2r$ centered at x.

We leave it to the reader to check that these two bases, in fact, give the same topology on $\mathbb{R}^2$ (hint: a square centered at x contains a ball centered at x which itself contains a square centered at x.)

Topologies are defined in terms of open subsets. There is a complementary notion of a closed subset for a topology.

Definition 1.1.10. Let X be a space. A subset $A \subseteq X$ is *closed* if $X \setminus A$ is open.

Example 1.1.11. If we take $X = \mathbb{R}$ with the standard topology, then $[a, b]$, $[a, +\infty)$, $(-\infty, +\infty)$, and $\{a\}$ are all examples of closed subsets.

Given that closed subsets are complements of open subsets, the axioms in Definition 1.1.1 can be reinterpreted in terms of closed subsets.

Lemma 1.1.12. *Let $(X, \mathcal{O})$ be a topological space.*

(1) The subsets $\varnothing$ and X are closed subsets.

(2) Arbitrary intersections of closed subsets gives a closed subset.

(3) Finite unions of closed subsets gives a closed subset.

Proof. We verify each claim in turn.

(1) Since $\varnothing$ is open, we have that $X = X \setminus \varnothing$ is closed. Similarly, since X is open, we have that $\varnothing = X \setminus X$ is closed.

(2) Write $A_\alpha = X \setminus U_\alpha$ for some arbitrary open subsets U_α. By de Morgan's law, we have that

$$\cap_\alpha A_\alpha = \cap_\alpha (X \setminus U_\alpha) = X \setminus \cup_\alpha U_\alpha.$$

However, $\cup_\alpha U_\alpha$ is open. So $\cap_\alpha A_\alpha$ is closed.

(3) Write $A_i = X \setminus U_i$ for some finite number of open subsets U_i. By de Morgan's law, we have that

$$\cup_{i=1}^n A_i = \cup_{i=1}^n (X \setminus U_i) = X \setminus \cap_{i=1}^n U_i.$$

However, $\cap_{i=1}^n U_i$ is open. So $\cup_{i=1}^n A_i$ is closed.

□

1.2 Interiors and closures of subsets

In real analysis, one eventually considers the interiors and closures of subsets in $\mathbb{R}^n$ or more general metric spaces. These notions have generalizations for arbitrary topological spaces.

Definition 1.2.1. Let $(X, \mathcal{O})$ be a topological space and let $A \subset X$ be a subset.

(1) The *interior* of A is the open subset given by

$$\text{int}(A) = \bigcup_{U \in \mathcal{O} \text{ s.t. } U \subset A} U.$$

(2) The *closure* of A is the closed subset given by

$$\overline{A} = \bigcap_{C \text{ closed s.t. } A \subset C} C.$$

Put into words, the interior of a subset A is the union of all open subsets that are contained in A. The closure of a subset A is the intersection of all closed subsets that contain A. In particular, $\text{int}(A)$ is the largest open subset that is contained inside of A and $\overline{A}$ is the smallest closed subset that contains all of A.

Example 1.2.2. (1) Consider a set X with the discrete topology. Any subset $A \subseteq X$ is open and closed. So $\overline{A} = A = \text{int}(A)$.

(2) Consider a set X with the trivial topology. Any non-empty proper subset $A \subsetneq X$ has $\overline{A} = X$ and $\text{int}(A) = \varnothing$.

Our notions of interiors and closures actually give characterizations of open and closed subsets.

Lemma 1.2.3. *Let X be a space and let A be a subset of X.*

(1) A is open if and only if $A = \text{int}(A)$.

(2) A is closed if and only if $A = \overline{A}$.

Proof. We verify each item in turn.

(1) If A is open, then $A \subseteq A$ is an open subset contained in A. Consequently, $\text{int}(A) = A$. Conversely, if $\text{int}(A) = A$, then since $\text{int}(A)$ is open, being a union of open subsets, we have that A is open.

(2) If A is closed, then $A \subseteq A$ is a closed subset containing A. Consequently, $\overline{A} = A$. Conversely, if $\overline{A} = A$, then since $\overline{A}$ is closed, being and intersection of closed subsets, we have that A is closed.

□

The notion of the closure gives rise to a notion of when a subset is dense.

Definition 1.2.4. Let X be a space. A subset $A \subseteq$X is *dense* if $\overline{A} = X$.

Example 1.2.5. Consider $\mathbb{R}$ with the standard topology. The subset $\mathbb{R} \setminus \mathbb{N}$ is a dense subset of $\mathbb{R}$. Similarly, $\mathbb{Q} \subset \mathbb{R}$ is also a dense subset. The fact that both of these subsets are dense can be derived using the relationship between the closure of a subset and the limit points of the subset. See the first conclusion of Lemma 1.2.8.

Remark 1.2.6. As a general warning, the intersections of dense subsets need not necessarily be dense. For example, consider $\mathbb{R}$ with the standard topology. The subsets $\mathbb{Q} \subset \mathbb{R}$ and $\mathbb{Q} + \sqrt{2} \subset \mathbb{R}$ are both dense in $\mathbb{R}$; however, their intersection is empty, which is not dense.

As hinted at above, to understand closures of subsets it is helpful to introduce the notion of a limit point of a subset.

Definition 1.2.7. Let X be a space and let $A \subset X$ be a subset.

(1) A point $x \in X$ is a *limit point* of A if for all opens U that contain x, we have that $A \cap U \neq \varnothing$.

(2) The *boundary* of A is the subset

$$\partial(A) = \{x \in X \mid x \text{ is a limit point of } A \text{ and } X \setminus A\}.$$

We have the following characterization of $\overline{A}$ in terms of limit points and boundary points.

Lemma 1.2.8. *Let X be a space and let A be a subset of X.*

(1) $\overline{A} = \{\textit{limit points of } A\} = \operatorname{int}(A) \cup \partial(A)$.

(2) $X = \operatorname{int}(A) \sqcup \partial(A) \sqcup \operatorname{int}(X \setminus A)$.

Proof. The proof of the second item is given in the exercises. So will will focus on the first item. For convenience, let $\mathcal{L}$ denote the set of limit points of the set A. We will show that

$$\overline{A} \subseteq \mathcal{L} \subseteq \operatorname{int}(A) \cup \partial(A) \subseteq \overline{A},$$

which will prove the desired statement. We handle each of these containments in turn:

- Fix $x \in \overline{A}$ and suppose by way of contradiction that $x \notin \mathcal{L}$. Then there exists an open subset U such that $x \in U$ and $A \cap U = \varnothing$. It follows that $A \subseteq (X \setminus U)$. But $X \setminus U$ is closed. So we have that $\overline{A} \subseteq X \setminus U$. But this implies that $x \notin \overline{A}$, a contradiction.

- Fix $x \in \mathcal{L}$. For each open subset U such that $x \in U$, we have that $A \cap U \neq \varnothing$. So if there exists an open subset U such that $x \in U \subseteq A$, then $x \in \operatorname{int}(A)$. If there does not exist such an open U, then for all open subsets U that contain x, we have that $U \cap (X \smallsetminus A) \neq \varnothing$. It follows that $x \in \partial(A)$.

- First, if $x \in \operatorname{int}(A)$, then $x \in \overline{A}$ since $\operatorname{int}(A) \subseteq A \subseteq \overline{A}$. So suppose that $x \in \partial(A)$ and suppose by way of contradiction that $x \notin \overline{A}$. It follows that $x \in X \smallsetminus \overline{A}$, which is open. Since $x \in \partial(A)$ and $X \smallsetminus \overline{A}$ is open, we have that

$$\varnothing \neq (X \smallsetminus \overline{A}) \cap A \subseteq (X \smallsetminus A) \cap A = \varnothing,$$

a contradiction.

□

As we noted above, the intersection of two dense subsets is not necessarily dense; however, if one of the dense subsets is open, then the intersection is dense. The proof of this claim utilizes the fact that Lemma 1.2.8 implies that a subset $A \subseteq X$ is dense if and only if it meets every non-empty open subset in X.

Lemma 1.2.9. *Let X be a space. Let $U \subseteq X$ be a dense open subset and let $A \subseteq X$ be a dense subset. The intersection $U \cap A$ is dense in X.*

Proof. Notice that by Lemma 1.2.8, we have that a subset is dense if and only if every point in X is a limit point of the subset. This in turn implies that a subset is dense if and only if it non-trivially intersects every non-empty open subset of X. So to show that $U \cap A$ is dense, it suffices to show that $V \cap (U \cap A)$ is non-empty for any open subset V. Since U is dense, $V \cap U \neq \varnothing$. Moreover, $V \cap U$ is open since V and U are open. Since A is dense, we have that $V \cap (U \cap A) = (V \cap U) \cap A \neq \varnothing$. This gives the desired result. □

1.3 Examples of topologies

In this section, we discuss several prototypical examples of topologies: metric topologies, subspace topologies, product topologies, and quotient topologies.

1.3.1 Metric spaces

In this subsection, we discuss sets with metrics. The existence of a metric on a set gives rise to a metric topology on the set. We begin by reviewing the definition of a metric.

Definition 1.3.1. A *metric* on a set X is a function $d\colon X \times X \to \mathbb{R}$ such that

(1) $d(x, y) \geq 0$ with $d(x, y) = 0$ if and only if $x = y$,

(2) $d(x, y) = d(y, x)$, and

(3) $d(x, z) \leq d(x, y) + d(y, z)$.

We also define the *ball of radius r centered at $x \in X$* as

$$B_x(r) = \{y \in X \mid d(x, y) < r\}.$$

Example 1.3.2. Let $X = \mathbb{R}^n$ and define

$$d(x, y) = \|x - y\|^2 = \left(\sum_{i=1}^{n} |x_i - y_i|^2 \right)^{1/2},$$

where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. The function d defines a metric on $\mathbb{R}^n$. We omit the proof of this fact here. An argument is given in most real analysis courses.

Definition 1.3.3. Let X be a set with a metric d. The *metric topology* for the pair (X, d) is the topology generated by the basis

$$\mathcal{B} = \{B_x(r) \mid x \in X, \, r \in \mathbb{R}_{>0}\}.$$

It remains to check that the above basis is, in fact, a basis.

Lemma 1.3.4. *The set of subsets $\mathcal{B}$ defined in Definition 1.3.3 is a basis.*

Proof. We verify the two axioms of a basis in turn.

(1) Notice that $B_x(1) \in \mathcal{B}$ for all $x \in X$. It follows that

$$X \subseteq \cup_{x \in X} B_x(1) \subseteq \cup_{B \in \mathcal{B}} B \subseteq X.$$

In particular, the elements of $\mathcal{B}$ cover X.

(2) Let $x \in B_{x_0}(r_0) \cap B_{x_1}(r_1)$. Let $\ell = \min(r_0 - d(x_0, x), r_1 - d(x_1, x))$. Notice that if $z \in B_x(\ell)$, then

$$d(x_0, z) \leq d(x_0, x) + d(x, z) \leq d(x_0, x) + r_0 - d(x_0, x) = r_0.$$

So $z \in B_{x_0}(r_0)$. Similarly, $z \in B_{x_1}(r_1)$. It follows that $x \in B_x(\ell) \subseteq B_{x_0}(r_0) \cap B_{x_1}(r_1)$. This verifies the second axiom.

□

Example 1.3.5. The *standard topology* on $\mathbb{R}^n$ is the metric topology on $\mathbb{R}^n$ with the metric

$$d(x, y) = \left(\sum_{i=1}^{n} |x_i - y_i|^2 \right)^{1/2}.$$

Remark 1.3.6. We warn the reader that different metrics on the same set can sometimes give rise to different topologies and can sometimes gives rise to the same topology. This is discussed further in the exercises.

1.3.2 Subspaces

Given a topological space X and a subset $A \subseteq X$, A can be endowed with a topology by using the topology of X. This is the subspace topology.

Definition 1.3.7. Let X be a space with a topology $\mathcal{O}$ and let $A \subseteq X$ be a subset. The *subspace topology* on A is

$$\mathcal{O}_A = \{A \cap U \mid U \in \mathcal{O}\}.$$

We call A with this topology a *subspace* of $(X, \mathcal{O})$.

It remains to check that $\mathcal{O}_A$ above actually defines a topology.

Lemma 1.3.8. *The set of subsets $\mathcal{O}_A$ defined in Definition 1.3.7 defines a topology on the set A.*

Proof. We verify the axioms of a topology in turn.

(1) Note that $\varnothing = A \cap \varnothing$ and $A = A \cap X$. So $\varnothing, A \in \mathcal{O}_A$.

(2) Let $V_\alpha \in \mathcal{O}_A$. So $V_\alpha = A \cap U_\alpha$ for some open subsets $U_\alpha \in \mathcal{O}$. Notice that

$$\bigcup_\alpha V_\alpha = \bigcup_\alpha (A \cap U_\alpha) = A \cap \bigcup_\alpha U_\alpha,$$

which is in $\mathcal{O}_A$ since $\cup_\alpha U_\alpha$ is open. This proves that an arbitrary union of opens is open.

(3) Let $V_i \in \mathcal{O}_A$ with $V_i = A \cap U_i$ for some $U_i \in \mathcal{O}$. Notice that

$$\bigcap_{i=1}^n V_i = \bigcap_{i=1}^n (A \cap U_i) = A \cap \bigcap_{i=1}^n U_i,$$

which is in $\mathcal{O}_A$ since $\cap_{i=1}^n U_i$ is open. This proves that a finite intersection of opens is open.

□

Remark 1.3.9. As an upshot of Definition 1.3.7, we have that *any* subspace in $\mathbb{R}^n$ is a topological space, endowed with the subspace topology. Here we take the standard topology on $\mathbb{R}^n$.

Just as the topology on X gives rise to a topology on a subset $A \subseteq X$. A basis for the topology of X also gives rise to a basis for the subspace topology of A.

Lemma 1.3.10. *Let $\mathcal{B}$ be a basis for a topological space $(X, \mathcal{O})$ and let $A \subseteq X$ be a subspace. The set of subsets*

$$\mathcal{B}_A = \{A \cap B \mid B \in \mathcal{B}\}$$

give a basis for the subspace $(A, \mathcal{O}_A)$.

Proof. There are two items that we need to check. First, we must show that $\mathcal{B}_A$ is, in fact, a basis. Next, we must check that the topology generated by this basis is the subspace topology of the subset A.

First let's show that $\mathcal{B}_A$ is a basis. Since the sets in $\mathcal{B}$ cover X, we have that the sets in $\mathcal{B}_A$ cover A. Next, suppose that $x \in (A \cap B') \cap (A \cap B'')$ is in the intersection of two basis elements. Then $x \in B' \cap B''$. So there exists $B \in \mathcal{B}$ such that $x \in B \subseteq B' \cap B''$. It follows that

$$x \in A \cap B \subseteq A \cap (B' \cap B'') = (A \cap B') \cap (A \cap B'').$$

This completes the check that $\mathcal{B}_A$ is a basis.

Now we need to check that the topology generated by $\mathcal{B}_A$ is the subspace topology of the subset A. To this end, suppose that $V \in \mathcal{O}_A$. So $V = A \cap U$ for some $U \in \mathcal{O}$. Since $\mathcal{B}$ generates $\mathcal{O}$, for each $x \in U$, we have that there exists $B \in \mathcal{B}$ such that $x \in B \subset U$. It follows that $x \in A \cap B \subseteq A \cap U = V$. So V is open in the topology generated by $\mathcal{B}_A$. Conversely, suppose that V is open in the topology generated by $\mathcal{B}_A$. This implies that for all $x \in V$, there exists $B_x \in \mathcal{B}$ such that $x \in A \cap B_x \subseteq V$. So we can write $V = \cup_x A \cap B_x = A \cap (\cup_x B_x) \in \mathcal{O}_A$. So V is open in $\mathcal{O}_A$. This proves that the two topologies agree. □

Remark 1.3.11. If X is a metric space with metric d and $A \subseteq X$, then A can be endowed with the structure of a metric space by defining a metric $d_A\colon A \times A \to \mathbb{R}$ by $d_A(x, y) = d(x, y)$.

Notice that if $A \subseteq X$ is a subset of a metric space X, then we can give A two different topologies. The first is the metric topology associated to the metric d_A. The second is the subspace topology of A. Perhaps unsurprisingly, these two topologies are the same.

Lemma 1.3.12. *Let X be a metric space with metric d and let $A \subseteq X$ be a subset. The metric topology on A with metric d_A in Remark 1.3.11 agrees with the subspace topology of A as a subset of X.*

Proof. To prove the claim, we need to show that a subset that is open in the metric topology is open in the subspace topology and vise-versa. First, note that the metric topology of X has basis

$$\mathcal{B} = \{B_x(r) \mid x \in X,\, r \in \mathbb{R}_{>0}\}.$$

It follows from Lemma 1.3.10 that a basis for the subspace topology of A is given by

$$\mathcal{B}_S = \{A \cap B_x(r) \mid x \in X,\, r \in \mathbb{R}_{>0}\}.$$

The metric topology on A has basis given by

$$\mathcal{B}_M = \{B^A_x(r) \mid x \in A,\, r \in \mathbb{R}_{>0}\} = \{A \cap B_x(r) \mid x \in A,\, r \in \mathbb{R}_{>0}\},$$

where $B_x^A(r)$ denotes the metric ball in A. It follows that $\mathcal{B}_M \subseteq \mathcal{B}_S$. Consequently, if a subset is open in the metric topology, then it is open in the subspace topology.

We now need to prove the opposite inclusion. To this end, fix an open subset $U \subseteq A$ in the subspace topology and fix $x \in U$. There exists $y \in X$ and $r \in \mathbb{R}_{>0}$ such that $x \in A \cap B_y(r) \subset U$. For $0 < \varepsilon \ll r$ sufficiently small, we have that $B_x(\varepsilon) \subset B_y(r)$. It follows that $A \cap B_x(\varepsilon) = B_x^A(\varepsilon) \subset U$. This proves that U is open in the metric topology, as desired. □

The subspace topology gives rise to the notion of a discrete subspace.

Definition 1.3.13. Let X be a space and let $A \subseteq X$ be a subset. The subset A is *discrete* if its subspace topology is the discrete topology.

Example 1.3.14. We give three simple examples of (non-)discrete subspaces. We will consider $X = \mathbb{R}$ with the standard topology.

(1) The subset $A = \{1/n \mid n \in \mathbb{N}\} \cup \{0\}$ is *not* discrete (even though it is made up of a countable number of points). To see that it is not discrete, we show that $\{0\}$ is not open in the subspace topology of A. To this end, suppose by way of contradiction that $\{0\}$ is open. This implies that there exists an open interval $(-\varepsilon, \varepsilon)$ such that $(-\varepsilon, \varepsilon) \cap A = \{0\}$. But for each $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that $1/n < \varepsilon$. Consequently, $(-\varepsilon, \varepsilon) \cap A \neq \{0\}$, a contradiction.

(2) Any finite collection of ponts in $\mathbb{R}$ is a discrete subspace. Indeed, suppose that $A = \{x_1, \ldots, x_n\}$. Since there are a finite number of points, we may find $\varepsilon > 0$ such that $x_j \notin (x_i - \varepsilon, x_i + \varepsilon)$ whenever $i \neq j$. It follows that $x_i = A \cap (x_i - \varepsilon, x_i + \varepsilon)$. Consequently, every point in A is an open subset and A has the discrete topology.

(3) The rational numbers $\mathbb{Q}$ are not a discrete subspace of $\mathbb{R}$. A similar argument as in the first example shows this.

We end this subsection by giving some warnings.

Remark 1.3.15. Let X be a space and let $A \subseteq X$ be a subspace. If $B \subset A$ is open in A, then it does not necessarily follows that $B \subset X$ is open. To see this, consider $X = \mathbb{R}^2$ with the standard topology and $A = \mathbb{R} \times \{0\}$ with the subspace topology. Taking $B = A$, we have that B is open in A; however, B is not open in X. Similarly, if B is closed in A, then B is not necessarily closed in X.

There is some saving grace to the above warning.

Lemma 1.3.16. *Let X be a space and let $A \subseteq X$ be a subspace.*

(1) If A is an open subset of X and $B \subseteq A$ is an open subset of A, then $B \subseteq X$ is an open subset of X.

(2) If A is a closed subset of X and $B \subseteq A$ is a closed subset of A, then $B \subseteq X$ is a closed subset of X.

The proof of the above lemma is an exercise.

Remark 1.3.17. Let X be a space and let $A \subseteq X$ and $B \subseteq A$ be subsets. The interior of B in A does not necessarily equal the intersection of the interior of B in X with A. To see this, take $X = \mathbb{R}^2$, $A = \mathbb{R} \times \{0\}$, and $B = (a, b) \times \{0\}$. Observe that the interior of B in X is empty. The interior of B in A is simply B.

Even though interiors do not behave nicely with respect to subspaces, closures do.

Lemma 1.3.18. *Let X be a space and let $A \subseteq X$ and $B \subseteq A$ be subsets. The closure of B in A is equal to the intersection of A with the closure of B in X.*

The proof of the above lemma is an exercise.

1.3.3 Product spaces

Given two topological spaces X and Y, we can take their Cartesian product as sets, $X \times Y$. This Cartesian product obtains a topology from the topologies of X and Y.

Definition 1.3.19. Let X and Y be spaces. The *product topology* on $X \times Y$ is the topology with basis

$$\{U \times V \mid U \subseteq X \text{ is open and } V \subseteq X \text{ is open}\}.$$

We need to show that the above definition is well-defined. That is, the claimed basis is, in fact, a basis.

Lemma 1.3.20. *The collection of subsets in Definition 1.3.19 forms a basis.*

Proof. We verify the axioms of a basis in turn.

(1) Since $X \subseteq X$ and $Y \subseteq Y$ are open subsets, we have that $X \times Y$ is in the collection of subsets. Consequently, it covers the set $X \times Y$.

(2) Suppose that $z \in (U_0 \times V_0) \cap (U_1 \times V_1)$ is a point in the intersection of two basic opens. Notice that

$$(U_0 \times V_0) \cap (U_1 \times V_1) = (U_0 \cap U_1) \times (V_0 \cap V_1),$$

which is also a basis open. So the second axion of a basis can always be satisfied.

□

Remark 1.3.21. Using Definition 1.3.19, one can inductively define a topology on finite Cartesian products of spaces: $X_1, \ldots, X_n$.

Remark 1.3.22. Let X and Y be spaces and let $\mathcal{B}$ and $\mathcal{C}$ be bases for the topologies of X and Y respectively. The Cartesian product $\mathcal{B} \times \mathcal{C}$ is a basis for the product topology of $X \times Y$. Indeed, let $\mathcal{A}$ denote the basis for the product topology that is given in Definition 1.3.19. We have that $\mathcal{B} \times \mathcal{C} \subseteq \mathcal{A}$. So the topology generated by the basis $\mathcal{B} \times \mathcal{C}$ is coarser than the topology generated by $\mathcal{A}$. Now let U be open in the product topology. Fix $x \in U$. So there exists opens $V \subseteq X$ and $W \subseteq Y$ such that $x \in V \times W \subset U$. Writing $x = (x_V, x_W)$ in coordinates, we have that there exist basis opens $A \in \mathcal{B}$ and $B \in \mathcal{C}$ such that $x_V \subseteq A \subseteq V$ and $x_W \subseteq B \subseteq W$. So $x \in A \times B \subset U$. It follows that U is open in the topology generated by the basis $\mathcal{B} \times \mathcal{C}$. This shows that the two topologies agree.

Notice now that we have two possibly different topologies on $\mathbb{R}^n$. The product topology and the metric topology. These two topologies agree.

Lemma 1.3.23. *The standard (metric) topology on $\mathbb{R}^n$, denoted $\mathcal{O}_M$, agrees with the product topology on $\mathbb{R}^n$, denoted $\mathcal{O}_P$.*

Proof. A subset $U \in \mathcal{O}_M$ if and only if for all $x \in U$, there exists $B_x(r) \subseteq U$. This occurs if and only if for all $x \in U$,

$$B_x(\varepsilon) \subseteq (x_1 - \varepsilon, x_1 + \varepsilon) \times \cdots \times (x_n - \varepsilon, x_1 + \varepsilon) \subseteq B_x(r)$$

for some $\varepsilon > 0$. But this occurs if and only if $U \in \mathcal{O}_P$. This shows that U is open in the metric topology if and only if it is open in the product topology, our desired result. □

Finally, given two spaces X and Y and two subsets $A \subseteq X$ and $B \subseteq Y$, we have two different topologies on $A \times B$. The subspace topology and the product topology associated to their respective subspace topologies. Again (hopefully the pattern is clear), these to topologies agree.

Lemma 1.3.24. *Let X and Y be spaces with subspaces $A \subseteq X$ and $B \subseteq Y$. The subspace topology of $A \times B \subset X \times Y$, denoted $\mathcal{O}_S$, agrees with the product topology of $A \times B$, denoted $\mathcal{O}_P$.*

Proof. To show that the two topologies agree, we show that they are both generated by the same basis. The topology $\mathcal{O}_S$ is generated by subsets of the form $(A \times B) \cap (U \times V)$ with $U \subseteq X$ and $V \subseteq Y$ open subsets. The topology $\mathcal{O}_P$ is generated by subsets of the form $(A \cap U) \times (B \cap V)$. However,

$$(A \times B) \cap (U \times V) = (A \cap U) \times (B \cap V).$$

So their bases agree. □

1.3.4 Quotient spaces

Given a space X with an equivalence relation $\sim$, we would like to use the topology on X to define a topology on the set of equivalence classes $X/\sim$. The quotient topology does this.

Definition 1.3.25. Let X be a space, let Y be a set, and let $q\colon X \to Y$ be a surjective map. The *quotient topology* on Y is determined via $U \subseteq Y$ is open if and only if $q^{-1}(U) \subseteq X$ is open.

We need to check that Definition 1.3.25 actual defines a topology on Y.

Lemma 1.3.26. *The open subsets in Definition 1.3.25 define a topology on* Y*.*

Proof. We verify the axioms of a topology in turn.

(1) Since q is surjective, we have that $q^{-1}(Y) = X$. So Y is open. Also, $q^{-1}(\varnothing) = \varnothing$. So $\varnothing$ is open.

(2) Suppose that $U_\alpha \subseteq Y$ are open subsets. Notice that

$$q^{-1}(\cup_\alpha U_\alpha) = \cup_\alpha q^{-1}(U_\alpha),$$

but the right hand side is open. So $\cup_\alpha U_\alpha$ is open.

(3) Suppose that $U_i \subseteq Y$ are open subsets for $i = 1, \ldots, n$. Notice that

$$q^{-1}(\cap_{i=1}^n U_i) = \cap_{i=1}^n q^{-1}(U_i),$$

but, again, the right hand side is open. So $\cap_{i=1}^n U_i$ is open.

□

Remark 1.3.27. Recall that an equivalence relation $\sim$ on a set X determines a surjection map $q\colon X \to X/\sim$. The quotient topology allows us to endow $X/\sim$, the set of equivalence classes, with a topology. Heuristically speaking, the quotient topology on $X/\sim$ describes the space that is obtained from X by "identifying" point in the same equivalence class with each other. Put differently, the quotient topology on $X/\sim$ describes the space that is obtained from X by "crushing" all equivalent points together to a single point.

Example 1.3.28. Let X be a space and let $A \subseteq X$ be a subset. We can define an equivalence relation $\sim_A$ on X via $x \sim_A y$ if and only if either $x = y$ or $x, y \in A$. We defined $X/A = X/\sim_A$ and endow it with the quotient topology from the surjective map $q\colon X \to X/A$. Heuristically speaking, X/A is the space that is obtained from X by crushing all points in A to a single point.

Example 1.3.29. For the below examples, we set

$$\mathbb{D}^2 = \left\{x \in \mathbb{R}^2 \|x\|^2 \leq 1\right\}.$$

This is the closed 2-dimension disk/ball.

(1) Consider $X = \mathbb{D}^2_1 \sqcup \mathbb{D}^2_2$, the disjoint union of two closed disks. We define an equivalence relation on X via $x \sim y$ if and only if either $x \in \partial\mathbb{D}^2_1$, $y \in \partial\mathbb{D}^2_2$ and $x = y$ (when viewed as points in $\mathbb{R}^2$) or $x = y$ in X. The quotient space is obtained by gluing together two unit disks along their boundaries. This produces a 2-dimensional sphere, S^2.

(2) Consider $X = \mathbb{D}^2$ and let $A = \partial\mathbb{D}^2$. The quotient space X/A is obtained by collapsing all points on the boundary of $\mathbb{D}^2$ to a single point. The resulting space is a 2-dimensional sphere, S^2.

(3) Consider $X = [0,1]^2$, a square. We define an equivalence relation on X via $(x_0, y_0) \sim (x_1, y_1)$ if and only if either the points are equal or $x_0 = 0$, $y_0 = y_1$, and $x_1 = 1$. The quotient space is obtained by gluing together two sides of the square to create a cylinder.

(4) Consider $X = [0,1]^2$. We define an equivalence relation on X via $(x_0, y_0) \sim (x_1, y_1)$ if and only if either the points are equal or $x_0 = 0$, $y_0 = y_1$, and $x_1 = 1$ or $y_0 = 0$, $x_0 = x_1$, and $y_1 = 1$. The quotient space is obtained by gluing together the horizontal sides of the square and then glueing together the vertical sides of the square. The resulting space is the 2-dimensional torus, T^2, an "intertube" if you will.

1.4 Continuous maps

In the past sections, we discussed topological spaces and examples there of. In this section, we discuss maps of topological spaces that preserve the topological structures. These are continuous maps.

1.4.1 Basics on continuous maps

Definition 1.4.1. Let X and Y be spaces. A map of sets $f\colon X \to Y$ is *continuous* if for each $U \subset Y$ open, $f^{-1}(U)$ is open.

Intuitively speaking, if we view open subsets as neighborhoods of points, then a map being continuous implies that if $f(x)$ and $f(x')$ are neighboring points, then x and x' are neighboring points. In other words, this is saying that "small changes" to the output correspond to "small changes" to the input. We will make this more precise later.

Just as topologies can be defined in terms of bases, we can use bases to capture the notion of continuity.

Lemma 1.4.2. *Let $f\colon X \to Y$ be a set map of spaces. Let $\mathcal{B}$ be a basis for the topology of Y. f is continuous if and only if $f^{-1}(B)$ is open for each $B \in \mathcal{B}$.*

Proof. Suppose that $U \subset Y$ is open, then by the exercises, $U = \cup_\alpha B_\alpha$ for some $B_\alpha \in \mathcal{B}$. So

$$f^{-1}(U) = f^{-1}(\cup_\alpha B_\alpha) = \cup_\alpha f^{-1}(B_\alpha).$$

But the right hand side is a union of open subsets by assumption. So we have that f is continuous. Conversely, if f is continuous, then $f^{-1}(B)$ is open for all $B \in \mathcal{B}$ since each B is open. □

Example 1.4.3. (1) Let X be a set. We consider both the discrete topology, denoted $\mathcal{O}_D$, and the trivial topology, denoted $\mathcal{O}_T$. We have the identity maps

$$\mathbb{1}\colon (X, \mathcal{O}_D) \longrightarrow (X, \mathcal{O}_T) \qquad \mathbb{1}\colon (X, \mathcal{O}_T) \longrightarrow (X, \mathcal{O}_D).$$

Notice that the map from the discrete topology to the trivial topology is continuous. However, the map from the trivial topology to the discrete topology is *not* continuous.

(2) Consider $\mathbb{R}$ endowed with the standard topology. Consider $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = 7 + x$. Notice that $f^{-1}((a, b)) = (a - 7, b - 7)$, which is open. So f is a continuous function.

The first property of continuity is that it is preserved by composition.

Lemma 1.4.4. *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be continuous maps of spaces. The composition $g \circ f\colon X \to Z$ is continuous.*

Proof. Let $U \subseteq Z$ be open. Since g is continuous, $g^{-1}(U)$ is open. Since f is continuous, $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ is open. This proves that the composition is continuous. □

Since continuous maps preserve the topological structures of spaces, we can use them to define when two spaces are equivalent as topological spaces (not just sets).

Definition 1.4.5. A continuous map of spaces $f\colon X \to Y$ is a *homeomorphism* if and only if it has a continuous inverse $g\colon Y \to X$. We say that X is *homeomorphic* to Y and write $X \cong Y$.

Remark 1.4.6. Two spaces are homeomorphic if and only if there is a bijection between their sets of points and if this bijection gives a bijection between their sets of open subsets. In other words, two spaces are homeomorphic if and only if they are the same set with the same topology (up to relabeling points).

To make our lives a little easier from this point forward, we will assume the following fact. A proof is typically given in a real analysis course and uses the equivalent epsilon-delta notion of continuity for metric spaces, see Corollary 1.4.20.

Fact 1.4.7. *(1) If $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ are continuous maps, then $(f+g)(x) = f(x)+g(x)$, $(f-g)(x) = f(x)-g(x)$, and $(f \cdot g)(x) = f(x) \cdot g(x)$ are continuous maps.*

(2) If $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ are continuous maps and $A \subseteq \mathbb{R}$ satisfies $g(x) \neq 0$ for all $x \in A$, then $(f/g)(x) = f(x)/g(x)$ defines a continuous map from A with the subspace topology to $\mathbb{R}$.

Example 1.4.8. We claim that $(-1,1)$ is homeomorphic to $\mathbb{R}$. We consider the maps $f\colon (-1,1) \to \mathbb{R}$ and $g\colon \mathbb{R} \to (-1,1)$ given by

$$f(x) = \frac{x}{1-|x|} \quad \text{and} \quad g(x) = \frac{x}{1+|x|}.$$

A computation shows that $f \circ g$ and $g \circ f$ are both the identity maps, that is, f and g are inverses of each other. So it remains to check that f and g are continuous. We will argue that f is continuous (a similar argument will show that g is continuous).

First, note that $h(x) = |x|$ is a continuous function since if (a,b) is an open interval, we have that

$$h^{-1}((a,b)) = \begin{cases} (a,b) \cup (-b,-a) & a \geq 0 \\ \varnothing & b \leq 0 \\ (-b,b) & \text{else} \end{cases},$$

which is open. So by Lemma 1.4.2, we have that $h(x) = |x|$ is continuous. Also notice that the constant function $c(x) = 1$ is continuous. Using item (1) of Fact 1.4.7, we have that $1+|x|$ is continuous on $(-1,1)$ and, moreover, using item (2), we have that f is continuous.

Remark 1.4.9. We warn the reader that if $f\colon X \to Y$ is a continuous, bijective map of space, then it does not necessarily follows that f is a homeomorphism. To see this, consider $X = \mathbb{R}$ with the discrete topology and $Y = \mathbb{R}$ with the standard topology. We take $f\colon X \to Y$ to be the identity map. Since X has the discrete topology, f is continuous. The inverse map $g\colon Y \to X$, which is the identity map, has $g^{-1}(0) = 0$. However, 0 is open in X, but it is not open in Y. Consequently, g is not continuous and f is not a homeomorphism.

Given a set map $f\colon X \to Y$, we can always restrict the map f to a subset $A \subseteq X$. Continuity is compatible with restriction in the following sense:

Lemma 1.4.10. *Let X be a space, let $A \subseteq X$ be a subspace, and let $f\colon X \to Y$ be a continuous map. The restriction $f|_A\colon A \to Y$ given by $f|_A(a) = f(a)$ is continuous.*

Proof. Let $U \subset Y$ be an open subset. Notice that $(f|_A)^{-1}(U) = A \cap f^{-1}(U)$. Since f is continuous, $f^{-1}(U)$ is open. Consequently, $(f|_A)^{-1}(U)$ is open in A. This shows that $f|_A$ is continuous. □

Corollary 1.4.11. *Let X be a space and let $A \subseteq X$ be a subspace. The inclusion map $i\colon A \to X$ given by $i(a) = a$ is continuous.*

Proof. The inclusion map is the restriction of the identity map to A. Since the identity map is continuous, Lemma 1.4.10 gives that the inclusion is continuous. □

We now dive into studying how continuity behaves with respect to the product topology. We begin with projection maps:

Lemma 1.4.12. *Let X and Y be spaces. The projection maps*

$$\mathrm{pr}_X\colon X \times Y \to X \qquad \mathrm{pr}_X(x, y) = x$$

and

$$\mathrm{pr}_Y\colon X \times Y \to Y \qquad \mathrm{pr}_Y(x, y) = y$$

are continuous.

Proof. We argue that pr_X is continuous. If $U \subset X$ is open, then $\mathrm{pr}_X^{-1}(U) = U \times Y$, which is open, as desired. □

The nice feature of the product topology is that a function to a product is continuous if and only if it is component-wise continuous. Typically checking component-wise continuity is much easier.

Lemma 1.4.13. *Consider a set map of spaces $f\colon Z \to X \times Y$ and write $f(z) = (f^X(z), f^Y(z))$, where $f^X : Z \to X$ and $f^Y : Z \to X$. The map f is continuous if and only if f^X and f^Y are continuous.*

Proof. Let $U \subseteq X$ and $V \subseteq Y$ be two open subsets and consider the basic open $U \times V \subset X \times Y$. Notice that

$$f^{-1}(U \times V) = (f^X)^{-1}(U) \cap (f^Y)^{-1}(V).$$

So if f^X and f^Y are continuous, then $(f^X)^{-1}(U)$ and $(f^Y)^{-1}(V)$ are open and, consequently, by Lemma 1.4.2, f is continuous.

Conversely, let us assume that f is continuous. Let $U \subseteq X$ be open. Notice that

$$f^{-1}(U \times Y) = (f^X)^{-1}(U) \cap Y = (f^X)^{-1}(U),$$

but the left hand side is open because f is continuous. Consequently, f^X is continuous. An analogous argument shows that f^Y is continuous. □

Corollary 1.4.14. *Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is a map of sets and write*

$$f(x) = (f_1(x), \ldots, f_n(x)),$$

where $f_i\colon \mathbb{R}^n \to \mathbb{R}$. f is continuous if and only if each f_i is continuous.

Example 1.4.15. Consider

$$S^n(r) \coloneqq \{x \in \mathbb{R}^{n+1} \mid \|x\| = r\}.$$

We claim that $S^n(1)$ is homeomorphic to $S^n(r)$ for every $r \in \mathbb{R}_{>0}$. Consider the maps

$$f\colon \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}, \qquad f(x) = r \cdot x$$

and

$$g\colon \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}, \qquad g(x) = x/r.$$

These maps are inverses of each other. We will show that f and g are continuous. One needs to show that the restrictions of f and g to $S^n(1)$ and $S^n(r)$ and to their images defines a homeomorphism. These remaining details will be left to the reader in the exercises.

We will argue that f is continuous (the same argument will show that g is continuous). First, notice that the function $h(x) = r \cdot x$ with r a fixed non-zero constant and $x \in \mathbb{R}$ is continuous. Indeed, the pre-image of an open interval is an open interval. Notice that $f_i(x_1, \ldots, x_n) = h \circ \mathrm{pr}_i(x_1, \ldots, x_n)$. So by Lemma 1.4.4 and Lemma 1.4.12, we have that f_i is continuous. By Corollary 1.4.14, we have that f is continuous, as desired.

Finally, we discuss continuity in the context of quotient spaces.

Definition 1.4.16. Let X be a space and let $q\colon X \to Y$ be a surjective map. Endow Y with the quotient topology. The map q is called the *quotient map*

Lemma 1.4.17. *The quotient map $q\colon X \to Y$ is continuous.*

Proof. Let $U \subset Y$ be open. By definition, this means that $q^{-1}(U)$ is open. Equivalently, q is continuous. □

We conclude with the following slight generalization of Lemma 1.4.17.

Lemma 1.4.18. *Given a quotient map $q\colon X \to Y$ and a set map of spaces $f\colon Y \to Z$, the map f is continuous if and only if $f \circ q$ is continuous.*

Proof. Fix an open subset $U \subset Z$. f is continuous if and only if $f^{-1}(U)$ is open for all $U \subseteq Z$ open. This is open if and only if $q^{-1}(f^{-1}(U))$ is open. This is true for all $U \subseteq Z$ open if and only if $q \circ f$ is continuous. This proves the desired statement. □

Our original definition of continuity is useful, as demonstrated above; however, at times, it is convenient to work with equivalent formulations of continuity. We give four equivalent formulations below.

Proposition 1.4.19. *Fix spaces X and Y. The following are equivalent.*

(1) $f\colon X \to Y$ is a continuous map.

(2) $f^{-1}(C)$ *is closed for all* $C \subseteq Y$ *closed.*

(3) For each $x \in X$ *and* $V \subseteq Y$ *open such that* $f(x) \in V$*, there exists an open subset* $U \subseteq X$ *such that* $f(U) \subseteq V$*.*

(4) For all subsets $A \subseteq X$*,* $f(\overline{A}) \subseteq \overline{f(A)}$*.*

Proof. We show that (1) holds if and only if (2) holds. Then we will show that (1) holds if and only if (3) holds. Then we will show that (2) holds if and only if (4) holds. This will show that the four conditions are logically equivalent.

- Let $C \subseteq Y$ be closed. So $C = Y \setminus U$ for $U \subseteq Y$ open. So

$$f^{-1}(C) = f^{-1}(Y \setminus U) = f^{-1}(Y) \setminus f^{-1}(U) = X \setminus f^{-1}(U).$$

  So if f is continuous, then $f^{-1}(C)$ is closed. Conversely, if $f^{-1}(C)$ is closed, then $f^{-1}(U)$ is open. This proves the equivalence of (1) and (2).

- Suppose that $f(x) \in V \subseteq Y$ with V open and set $U = f^{-1}(V)$, which is open in X. Then $f(U) = f(f^{-1}(V)) \subseteq V$. This show that (1) implies (3). Conversely, assume condition (3) and let $V \subseteq Y$ be open. For each x such that $f(x) \in f(X) \cap V$, there exists an open subset $U_x \subseteq X$ such that $f(U_x) \subseteq V$. Defining

$$U = \bigcup_{x \text{ s.t. } f(x) \in V} U_x,$$

  we have that $f^{-1}(V) = U$, which is open. Consequently, f is continuous. This show that (3) implies (1).

- Let $A \subseteq X$ be a subset. Notice that

$$A \subseteq f^{-1}(f(A)) \subset f^{-1}(\overline{f(A)}).$$

  By condition (2), the right hand side is closed, so $\overline{A} \subseteq f^{-1}(\overline{f(A)})$. Applying f to this inclusion gives $f(\overline{A}) \subseteq \overline{f(A)}$. This show that (2) implies (4). Conversely, if $C \subseteq Y$ is closed, then set $f^{-1}(C) = A \subseteq X$. By condition (4), we have that

$$f(\overline{A}) \subseteq \overline{f(f^{-1}(C))} \subseteq C.$$

  Apply f^{-1} to this chain, we have that

$$A \subseteq \overline{A} \subseteq f^{-1}(f(\overline{A})) \subseteq f^{-1}(C) = A.$$

  If follows that $A = f^{-1}(C) = \overline{A}$, which is closed, implying condition (2).

□

As a corollary of Proposition 1.4.19, we have the following well-known epsilon-delta formulation of continuity for metric spaces:

Corollary 1.4.20. *Let X and Y be metric spaces. A map $f\colon X \to Y$ is continuous if and only if for all $x \in X$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $d_X(x, x') < \delta$ implies that $d_Y(f(x'), f(x)) < \varepsilon$.*

Proof. We begin by assuming that f is continuous. To this end, fix $x \in X$ and $\varepsilon > 0$. By Proposition 1.4.19 item (3), we have that there exists $U \subseteq X$ open such that $x \in U$ and $f(U) \subseteq B_{f(x)}(\varepsilon)$. Since U is open, there exists $\delta > 0$ such that $B_x(\delta) \subseteq U$. It follows that $f(B_x(\delta)) \subseteq B_{f(x)}(\varepsilon)$; however, this is equivalent to the condition that we are trying to prove.

Now conversely, let us suppose that the epsilon-delta condition holds. Let $V \subseteq Y$ be open and fix $x \in X$ such that $f(x) \in V$. Since V is open, there exists $\varepsilon > 0$ such that $B_{f(x)}(\varepsilon) \subseteq V$. So by the condition, there exists $\delta > 0$ such that $f(B_x(\delta)) \subseteq B_{f(x)}(\varepsilon) \subseteq V$. But this proves item (3) of Proposition 1.4.19, proving the desired direction. □

To wrap up our preliminary discussion on continuity, we discuss a useful lemma for creating continuous maps by pasting together continuous maps on closed subsets.

Lemma 1.4.21. *[Pasting Lemma] Let X be a space and suppose that $X = A \cup B$, where A and B are closed subspaces of X. If $f_A\colon A \to Y$ and $f_B\colon B \to Y$ are continuous maps such that $f_A(x) = f_B(x)$ for all $x \in A \cap B$, then*

$$f\colon X \to Y, \qquad f(x) = \begin{cases} f_A(x) & x \in A \\ f_B(x) & x \in B \end{cases}$$

is continuous.

Proof. First, notice that the map f is well-defined since the ambiguous part of its definition is remedied by the fact that f_A equals f_B on the intersection. Now suppose that $C \subseteq Y$ is closed. We will show that $f^{-1}(C)$ is closed. Note, $f_A^{-1}(C) = f^{-1}(C) \cap A$, which is closed in A. Similarly, $f_B^{-1}(C) = f^{-1}(C) \cap B$ is closed in B. By the exercises, we have that $f_A^{-1}(C)$ is closed in X. Similarly, $f_B^{-1}(C)$ is closed in X. So $f^{-1}(C) = f_A^{-1}(C) \cup f_B^{-1}(C)$ is closed in X, as desired. □

Example 1.4.22. Consider $f\colon \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 0 & x \leq 0 \\ x & x \geq 0 \end{cases}.$$

By the Pasting Lemma, f is continuous.

Example 1.4.23. Write $I = [0,1]$. Let $\alpha\colon I \to X$ and $\beta\colon I \to X$ be continuous maps such that $\alpha(1) = \beta(0)$. Define

$$\alpha \star \beta\colon I \longrightarrow X, \qquad (\alpha \star \beta)(t) = \begin{cases} \alpha(2t) & 0 \leq t \leq 1/2 \\ \beta(2t-1) & 1/2 \leq t \leq 1 \end{cases}.$$

By the Pasting Lemma, $\alpha \star \beta$ is continuous. We call $\alpha \star \beta$ the *concatenation* of α and β.

1.4.2 Limits and continuity

A natural question to ask is whether or not a continuous map preserves some notion of limits. To make sense of such a question, we must make sense of limits for arbitrary topological spaces:

Definition 1.4.24. A sequence of points $x_n \in X$ converses to a point $x \in X$ if and only if for every open subset $U \subseteq X$ such that $x \in U$, the set U contains all but finitely many of the points x_n. We write $x_n \to x$.

Remark 1.4.25. In real analysis one observes that limits in $\mathbb{R}^n$ are always unique; however, this is not true in general. Limits need not necessary be unique. We can see this in two examples:

(1) Let X be a set with the trivial topology and set $x_n = x$ for all n and some $x \in X$. Notice that for all $y \in X$, we have that $x_n \to y$. Consequently, the limit is not unique.

(2) Consider $X = \mathbb{R}_1 \sqcup \mathbb{R}_2/\sim$ where $x \sim y$ if and only if either $x = y$ or $x \in \mathbb{R}_1 \smallsetminus \{0\}$, $y \in \mathbb{R}_2 \smallsetminus \{0\}$, and x equals y when both viewed as points in $\mathbb{R}$. This space is called the "fat point". Let $\mathfrak{o}_1$ and $\mathfrak{o}_2$ denote the images of the origins in $\mathbb{R}_1 \sqcup \mathbb{R}_2$ in X. Notice that they map to distinct points. Consider $x_n = [1/n]$, where $1/n \in \mathbb{R}_1$. Notice that $x_n \to \mathfrak{o}_1$ and $x_n \to \mathfrak{o}_2$. Consequently, the limit is not unique.

Lemma 1.4.26. *Let X be a space and let $A \subseteq X$ be a subset. If a sequence of points x_n is contained in A and $x_n \to x$, then $x \in \overline{A}$.*

Proof. We will use the limit point formulation of the closure to prove this result. Let U be an open subset that contains x. Since $x_n \to x$, $U \cap \cup_n \{x_n\} \neq \varnothing$. It follows that $U \cap A \neq \varnothing$. Since U was arbitrary, we have that $x \in \overline{A}$. $\square$

We warn the reader that the converse of Lemma 1.4.26 is not true. That is, if $x \in \overline{A}$, then this does not imply that there exists a sequence $x_n \in A$ such that $x_n \to x$. A counter-example is given in the exercises. To obtain the converse, we need to assume that our space X is first countable.

Definition 1.4.27. A space X is *first countable* if and only if for all $x \in X$ there exists a countable number of open subsets $U_1, \dots, U_n$ such that if U is an open subset that contains x, then $U_i \subseteq U$ for some i.

Example 1.4.28. Every metric space is first countable. For each x, one can consider the balls $B_x(1/n)$.

There is an equivalent formulation of first countability that is typically easier to work with.

Lemma 1.4.29. *A space X is first countable if and only if for all $x \in X$ there exists a countable number of open subsets $V_1 \supseteq V_2 \supseteq \cdots$ such that if U is an open subset that contains x, then $V_i \subset U$ for some i.*

Proof. Let $x \in X$ and let U_i be as in the definition of first countable. We define $V_1 = U_1$. We define V_m inductively as follows: $V_{m-1} \cap U_m$ is open and contains x. So there exists U_{n_m} such that $U_{n_m} \subseteq U_m \cap V_{m-1}$. We set $V_m = U_{n_m}$. This collection of subsets satisfies the conditions in the lemma. □

We now can prove the converse of Lemma 1.4.26 for first countable spaces.

Lemma 1.4.30. *Let X be first countable and let $A \subseteq X$ be a subset. If $x \in \overline{A}$, then there exist $x_n \in A$ such that $x_n \to x$.*

Proof. Consider open subsets $V_1, \dots, V_n, \dots$ as in Lemma 1.4.29. Since $x \in \overline{A}$, $V_i \cap A \neq \varnothing$. So there exists $x_i \in A \cap V_i$. Now let U be an open subset that contains x. Then there exists V_n such that $V_n \subseteq U$. This implies that $x_k \in U$ for all $k \geq n$. Consequently, U meets all but finitely many of the points x_n. It follows that $x_n \to x$, as desired. □

We now start to analyze how continuity of maps plays with limits of sequences. We have the following implication for all spaces.

Lemma 1.4.31. *Suppose $f\colon X \to Y$ is continuous. If $x_n \to x$, then $f(x_n) \to f(x)$.*

Proof. We will prove the contrapositive statement. Suppose that $f(x_n) \not\to x$. There exists an open U that contains $f(x)$ such that U does not contain infinitely many of the $f(x_n)$. This implies that $f^{-1}(U)$ does not contain infinitely many of the x_n. Indeed, if $f^{-1}(U)$ contained all but finitely many of the x_n, then all but finitely many of the x_n would be mapped via f into U, but then U would contain all but finitely many of the $f(x_n)$, a contradiction. So we find that $x_n \not\to x$. □

When the space X is first countable, the converse of Lemma 1.4.31 holds.

Lemma 1.4.32. *Let X be first countable and let $f\colon X \to Y$ be a set map of spaces. If for all $x \in X$ and all sequences $x_n \to x$ one has that $f(x_n) \to f(x)$, then f is continuous.*

Proof. We will show that $f(\overline{A}) \subset \overline{f(A)}$ for an arbitrary subset A, which is equivalent to f being continuous. So fix $A \subseteq X$. Let $x \in \overline{A}$ and $x_n \in A$ be a sequence such that $x_n \to x$. Such a sequence exists by Lemma 1.4.30. By Lemma 1.4.26, $f(x_n) \to f(x)$ implies that $f(x) \in \overline{f(A)}$. This implies that $f(\overline{A}) \subset \overline{f(A)}$, which is what we wanted to show. □