18.901– Introduction to topology

Midterm 2

MIT

Instructor: Alex Pieloch

11/14/23

Name: ________________________________________________________________

Student Number: ______________________________________________________

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• This exam contains 17 pages and 13 questions.

• This exam is out of 56 points. The distribution of points among all of the questions is shown in the table on the page 2 and is also indicated next to each question.

• You will have 80 minutes to complete the exam.

• Good luck!
## Distribution of Marks

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1 Definitions and statements

1. (3 points) Give the definition of a partition of unity on a space $X$.

2. (2 points) Given two continuous maps $f: X \to Y$ and $g: X \to Y$ and a subspace $A \subset X$, define what it means for $f$ to be homotopic to $g$ rel $A$.

3. (2 points) Define what it means for two spaces $X$ and $Y$ to be homotopy equivalent.

4. (3 points) State the Lebesgue covering lemma.
2 True/False

For Questions 5-8, state whether or not the given statement is true or false. If it is true, provide a proof of the statement. If it is false, provide either a counter-example to the statement or a disproof.

5. (4 points) For every manifold $X$ there exists a countable number of open subsets $U_i \subset X$ such that $U_i \cong \mathbb{R}^n$ and $X = \bigcup_i U_i$.

6. (3 points) A group homomorphism is injective if and only if its kernel is the trivial subgroup.
7. (4 points) Let $X$ be a space with covering dimension equal to $n$. If $A \subset X$ is a closed subspace, then $A$ has covering dimension less than or equal to $n$.

8. (4 points) If $\alpha : [0,1] \to X$ and $\beta : [0,1] \to X$ are homotopic rel $\partial [0,1]$ and $\gamma : [0,1] \to X$ satisfies $\alpha(1) = \gamma(0)$, then $\alpha \cdot \gamma$ is homotopic to $\beta \cdot \gamma$ rel $\partial [0,1]$. 
3 Free response

9. (6 points) Let $X$ be a manifold and fix $x_0 \in X$. Show that there exists a continuous map $f: X \to \mathbb{R}^N$ for some $N > 0$ and an open neighborhood $U$ of $x_0$ that satisfy: For all $x \in U$, if $f(x) = f(y)$, then $x = y$. 
10. (6 points) Let $X$ be a paracompact space. Let $x \in X$ and let $A \subset X$ be a closed subset that does not contain $x$. Show that there exist open sets $U, V \subset X$ such that $x \in U$, $A \subset V$, and $U \cap V = \emptyset$. 
11. Consider the map \( \varphi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}/10\mathbb{Z} \) given by \( \varphi(x, y) = [2x + 3y] \).

   (a) (2 points) Show that \( \varphi \) is a homomorphism.

   (b) (1 point) What is the image of \( \varphi \)?
(c) (2 points) Show that \( H = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x = \frac{10k - 3y}{2} \text{ for some } k \in \mathbb{Z}\} \) is a normal subgroup.

(d) (2 points) Describe the quotient group: \((\mathbb{Z} \times \mathbb{Z})/H\).
12. (6 points) Consider the CW-complex $X$ that is constructed as follows: There is a single 0-cell $v$ in $X^0$. There are three 1-cells in $X^1$ that are attached to $v$ by the unique map $\partial D^1 \to \{v\}$. This gives a wedge of three circles. Parameterize each of these circles by curves $\alpha$, $\beta$, $\gamma$ respectively.

There are three 2-cells in $X^2 = X$ that are attached via the maps $\varphi_i: S^1 \to X^1$ by

- $\varphi_1(s) = \alpha \cdot \beta \cdot \alpha^{-1} \cdot \beta^{-1} \cdot \gamma \cdot \gamma \cdot \gamma(s),$
- $\varphi_2(s) = \alpha(s),$ and
- $\varphi_3(s) = \alpha^{-1}(s).$

Let $A \subset X$ denote the subcomplex given by $\{v\} \cup \{\gamma\}$. Show that $X/A$ is homotopy equivalent to a wedge of familiar spaces. (You may draw pictures, but make sure to justify your pictures.)
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13. (6 points) Consider $S^1 \lor S^1$ with basepoint $x_0$ given by the intersection of the two copies of $S^1$. Show that there exist infinitely many non-homotopic rel \{x_0\} retractions of $S^1 \lor S^1$ onto one of the copies of $S^1$. 
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