18.901 - Homework 2

Fall 2023

Please read and follow the instructions below.

• Write and organize your solutions as clearly as possible. Marks may be deducted at the discretion of the grader for illegibility or poor writing.

• Working with fellow classmates and seeking help from others, if necessary, is highly encouraged, but collaboration and help should be acknowledged appropriately at the top of your solutions. The copying of answers from other students or outside sources is considered cheating.

• Given a function $f: \mathbb{R} \to \mathbb{R}$ that you know to be continuous from real analysis, you may assume without proof that $f$ is continuous here. For example, you may assume that polynomials, exp, log, square roots, etc. are continuous functions.
**Question 1** Let $X$ be a topological space and let $A \subset X$ be a subspace.

(i) Show that if $B \subset A$ is open in the subspace topology of $A$ and $A \subset X$ is open in $X$, then $B$ is open in $X$.

(ii) Show that if $B \subset A$ is closed in the subspace topology of $A$ and $A \subset X$ is closed in $X$, then $B$ is closed in $X$.

**Question 2** Let $X$ be a topological space and let $Y \subset X$ be a subspace. Let $A \subset Y \subset X$ be a subset. Show that the closure of $A$ in $Y$ with respect to the subspace topology of $Y$ is equal to the intersection of $Y$ with the closure of $A$ in $X$.

**Question 3** Give an example of a set $X$ and two different metrics $d: X \times X \to \mathbb{R}$ and $d': X \times X \to \mathbb{R}$ such that the metric topologies determined by $d$ and $d'$ do not agree. (Hint: The discrete topology.)

**Question 4** Let $X$ be a topological space. Show that if $f: X \to \mathbb{R}$ and $g: X \to \mathbb{R}$ are continuous functions, then $f \circ g$, $f \cdot g$, and $f + g$ are all continuous functions. Moreover, if $g \neq 0$, show that $f/g$ is a continuous function. (Hint: Can you reduce to the case of $X = \mathbb{R}$ with the standard topology and results from class?)

**Question 5** Let $f_1: X_1 \to Y_1$ and $f_2: X_2 \to Y_2$ be homeomorphisms. Show that

$$f_1 \times f_2: X_1 \times X_2 \to Y_1 \times Y_2, \quad f_1 \times f_2(x_1, x_2) = (f_1(x_1), f_2(x_2))$$

is a homeomorphism.

**Question 6** Let $X$ and $Y$ be spaces and let $f: X \to Y$ be a map. Define

$$f(X) = \{y \in Y \mid f(x) = y \text{ for some } x \in X\}.$$ Consider $f(X) \subset Y$ with the subspace topology. Show that $f$ is continuous if and only if $f: X \to f(X)$ is continuous.

**Question 7** Let $f: X \to Y$ be a homeomorphism. Let $A \subset X$ be a subset.

(i) Describe the closure, interior, and boundary of $f(A)$.

(ii) Endow $A$ with the subspace topology. Show that the restriction of $f$ to $A$ defines a homeomorphism from $A$ to $f(A)$, where $f(A) \subset Y$ is endowed with the subspace topology. (Hint: Question 6 above is helpful.)

**Question 8** Let

$$\mathbb{D}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 \leq 1\}$$

Endow $\mathbb{D}^n \subset \mathbb{R}^n$ with the subspace topology, where $\mathbb{R}^n$ is equip with the standard topology. Let

$$S^{n-1} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 = 1\}$$

(i) Show that $\partial \mathbb{D}^n = S^{n-1}$.

(ii) Show that $\text{int}(\mathbb{D}^n)$ is homeomorphic to $\mathbb{R}^n$.

**Question 9** Endow $\mathbb{R}^{n+1} \setminus (0, \ldots, 0) \subset \mathbb{R}^{n+1}$ with the subspace topology, where $\mathbb{R}^{n+1}$ is equip with the standard topology. Equip $S^n \times (0, +\infty)$ with the product topology. Show that $\mathbb{R}^{n+1} \setminus (0, \ldots, 0) \cong S^n \times (0, +\infty)$.

**Question 10** Define

$$C^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \in [-1, 1]\}.$$ (i) Show that $C^n$ is homeomorphic to $\mathbb{D}^n$.

(ii) Show that $\text{int}(C^n)$ is homeomorphic to $\text{int}(\mathbb{D}^n)$.

(iii) Show that $\partial C^n$ is homeomorphic to $S^{n-1}$.  

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