Please read and follow the instructions below.

- Write and organize your solutions as clearly as possible. Marks may be deducted at the discretion of the grader for illegibility or poor writing.

- Working with fellow classmates and seeking help from others, if necessary, is highly encouraged, but collaboration and help should be acknowledged appropriately at the top of your solutions. The copying of answers from other students or outside sources is considered cheating.

- Students should refrain from using language like: ”It is clear...”, ”obviously”, ”It is easy to show”, ”Clearly”, etc. in their solutions.

- Given a function \( f: \mathbb{R} \to \mathbb{R} \) that you know to be continuous from real analysis, you may assume without proof that \( f \) is continuous here. For example, you may assume that polynomials, \( \exp \), \( \log \), square roots, etc. are continuous functions.
Question 1. Show that the change-of-basepoint isomorphism $\Phi_\gamma$ only depends on the homotopy class of $\gamma$ rel $\partial I$. That is, if $\gamma$ and $\delta$ are two paths in $X$ from $x_0$ to $x_1$ and $\gamma$ is homotopic to $\delta$ rel $\partial I$, then $\Phi_\gamma = \Phi_\delta$.

Question 2. Let $X$ be a path-connected space. Show that $\pi_1(X)$ is abelian if and only if all change-of-basepoint isomorphisms $\Phi_\gamma$ only depend on the endpoints of $\gamma$.

Question 3. Let $G$ be a group. Two elements $g, g' \in G$ are conjugate if and only if there exists $h \in G$ such that $g = h \cdot g' \cdot h^{-1}$. Show that conjugation gives an equivalence relation on the group $G$. The conjugacy classes of $G$ are the equivalence classes of the conjugation equivalence relation.

Question 4. Let $X$ be a path-connected space. We can view $\pi_1(X, x_0)$ as the set of basepoint-preserving homotopy classes of maps $(S^1, 0) \to (X, x_0)$. Let $[S^1, X]$ denote the set of homotopy classes of maps $S^1 \to X$ with no basepoint preserving conditions. There is a map of sets $\Phi: \pi_1(X, x_0) \to [S^1, X]$.
   (i) Show that $\Phi$ is onto.
   (ii) Show that $\Phi([\alpha]) = \Phi([\beta])$ if and only if $[\alpha]$ and $[\beta]$ are conjugate in $\pi_1(X, x_0)$.
   (iii) Show that $\Phi$ induces a one-to-one correspondence between $[S^1, X]$ and conjugacy classes in $\pi_1(X, x_0)$.

Question 5. Let $X$ and $Y$ be path-connected spaces. Show that $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$.

Question 6. Show that every homomorphism $\mathbb{Z} \cong \pi_1(S^1) \to \pi_1(S^1) \cong \mathbb{Z}$ can be realized as an induced homomorphism $f_*: \pi_1(S^1) \to \pi_1(S^1)$ for some continuous map $f: S^1 \to S^1$.

Question 7. Show that there are no retractions $r: X \to A$ in the following cases:
   (i) $X = \mathbb{R}^3$ and $A$ is a subspace that is homeomorphic to $S^1$.
   (ii) $X = S^1 \times D^2$ with $A = S^1 \times \partial D^2 \cong S^1 \times S^1$.
   (iii) $X = D^2 \vee D^2$ glued along the boundaries with $A = S^1 \vee S^1$, that is, $A$ is the boundary of $X$.
   (iv) $X$ is the space obtained from identifying two points on the boundary of a disk and $A$ is the boundary of $X$, which is given by $S^1 \vee S^1$.
   (v) $X$ is the Möbius band and $A$ is its boundary circle.