Lecture #9 - October 12th, 2023

Manifolds

Defn: A n-dim'l manifold X is a 2^{nd} countable Hausdorff space st $\forall x \in X$, $\exists$ open $x \in U$ and a homeo $\varphi_x : \mathbb{R}^n \longrightarrow U$.

↳ Typically call X an n-manifold and write X^n.

Ex: ① $\mathbb{R}^n$ = n-manifold

(i) $\mathbb{R}^n$ = metric space $\Rightarrow \mathbb{R}^n$ = Haus

(ii) $\mathbb{R}^n$ has basis $\{(a_1, b_1) \times \dots \times (a_n, b_n) \mid a_i, b_i \in \mathbb{Q}\} \Rightarrow 2^{nd}$ countable

(iii) $\forall x \in \mathbb{R}^n$, take $U = \mathbb{R}^n$

② $S^n = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$ = n-manifold

(i) S^n = subspace of Haus $\Rightarrow S^n$ = Haus

(ii) $\mathbb{R}^n = 2^{nd}$ countable $\Rightarrow S^n = 2^{nd}$ countable (Recall basis for subspace topology)

(iii) $H_i^{\pm} = \{(x_1, \dots, x_{n+1}) \in S^n \mid \pm x_i > 0\}$

$\overline{H_i^{\pm}} = \{(x_1, \dots, x_{n+1}) \in S^n \mid \pm x_i \geq 0\}$

Note $H_i^{\pm}$ cover S^n. STS that $H_i^{\pm} \cong \mathbb{R}^n$

Define $\varphi_i^{\pm} : \overline{B_0(1)} \longrightarrow \overline{H_i^{\pm}}$ by

$$\varphi_i^{\pm}(x_1, \dots, x_n) = \left(x_1, \dots, x_{i-1}, \pm\sqrt{1 - x_1^2 - \dots - x_n^2}, x_i, \dots, x_n\right)$$

$\varphi_i^{\pm}$ is cts and is bijective onto its image.

$\overline{B_0(1)}$ = cpt, $\overline{H_i^{\pm}}$ = Haus $\Rightarrow \varphi_i^{\pm}$ = homeo.

$\Rightarrow \varphi_i^{\pm}\big|_{B_0(1)} : B_0(1) \longrightarrow H_i^{\pm}$ is a homeo. □

③ $\mathbb{RP}^n = S^n/\sim$, $x \sim y$ iff $x = \pm y$

Write equiv class of x as $[x]$

$q: S^n \longrightarrow \mathbb{RP}^n$ is $\begin{cases} \text{open since } q^{-1}(q(U)) = U \cup -U = \text{open} \\ \text{closed } '' \quad q^{-1}(q(C)) = C \cup -C = \text{closed} \end{cases}$

$q^{-1}([x]) = 2 \text{ pts} = \text{cpt}$

(i) + (ii) follow from HW4 Exer 8.

Consider $\varphi_i^{\pm}: \mathbb{R}^n \longrightarrow H_i^{\pm}$

Define $\psi_i^{\pm}: \mathbb{R}^n \longrightarrow q(H_i^{\pm}) = \text{open}$, $\psi_i^{\pm} = q \circ \varphi_i^{\pm}$

$\psi_i^{\pm}$ is bijective + cts.

$\varphi_i^{\pm}, q$ are open $\Rightarrow \psi_i^{\pm} = \text{open}$

$\Rightarrow$ set-theoretic inv. is cts.

$\Rightarrow \psi_i^{\pm} = \text{homeo.}$

Since the $H_i^{\pm}$ cover S^n, the $q(H_i^{\pm})$ cover $\mathbb{RP}^n$. $\square$

Lemma: $X^n = n$-manifold, $Y^m = m$-manifold $\Rightarrow X^n \times Y^m = (n+m)$-manifold

Proof: (i) Prod. of Haus = Haus $\Rightarrow X \times Y = $ Haus

(ii) Basis of $X \times Y$ is $B_i \times B_j'$ where $B_i, B_j' = $ basis elm for X, Y resp.

So X, Y have countable bases, $X \times Y$ has countable basis.

(iii) Fix $(x,y) \in X \times Y$. $\exists$ opens $x \in U$, $y \in V$ and homeos

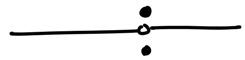
$\varphi: \mathbb{R}^n \to U$, $\psi: \mathbb{R}^m \to V \Rightarrow \varphi \times \psi: \mathbb{R}^n \times \mathbb{R}^m \longrightarrow U \times V$ homeo $\square$

Ex: $T^n = S^1 \times \ldots \times S^1 = n$-copies of $S^1 = n$-manifold.

$T^2 = $ Torus

Defn: A 1-manifold is a <u>curve</u>.

Fact: Every connected 1-manifold is homeo to either $\mathbb{R}$ or S^1.

Ex: ① Fat point :  $= \mathbb{R} \sqcup \mathbb{R}/\sim$

$=$ locally homeo to $\mathbb{R}$, 2^{nd} countable, but not Haus.

② $(\mathbb{R}, std) \times (\mathbb{R}, disc) =$ locally homeo to $\mathbb{R}$, Haus, but not 2^{nd} count.

Defn: An embedded n-manifold is a subspace $X \subseteq \mathbb{R}^N$ st $\forall x \in X$ $\exists$ open $x \in \mathcal{U} \subseteq X$ and a homeo $\varphi : \mathbb{R}^n \longrightarrow \mathcal{U}$.

Thm: $X =$ manifold iff $X =$ embedded manifold.

Proof: ($\Leftarrow$) X embedded $\Rightarrow$ $X =$ Haus $+$ 2^{nd} countable $\Rightarrow$ $X =$ manifold.

($\Rightarrow$) Will require work. $\square$

Lemma: $\exists$ a countable # of opens $\mathcal{U}_i \subseteq X$ st

① $\exists$ homeos $\varphi_i : \mathbb{R}^n \longrightarrow \mathcal{U}_i$

② $X = \bigcup_i \varphi_i(B_0(1))$

Proof: $\forall x$ $\exists$ $x \in \mathcal{U}_x \subseteq X$ open w/ $\varphi_x : \mathbb{R}^n \longrightarrow \mathcal{U}_x$ homeo $+$ $\varphi_x(0) = x$

$\exists$ basis element $x \in B_x \subseteq \varphi_x(B_0(1)) =$ open

2^{nd} count $\Rightarrow$ countable # of B_x's cover X

$\Rightarrow$ countable # of $\varphi_x(B_0(1))$ cover X $\square$

Proof: ($M =$ cpt manifold $\Rightarrow$ $M =$ emb manifold)

Above Lemma $+$ cpt $\Rightarrow$ $\exists$ $\mathcal{U}_1, \dots, \mathcal{U}_N$ st

① $\varphi_i : \mathbb{R}^n \longrightarrow \mathcal{U}_i$ homeo

② $\varphi_i(B(1))$ cover M.

Normal + Urysohn $\Rightarrow \exists\ \rho_i : M \to \mathbb{R}$ st

① $\rho_i(\varphi_i(\overline{B(1)})) \equiv 1$

② $\rho_i(M \setminus \varphi_i(B(2))) \equiv 0$

Defn $\psi_i : M \to \mathbb{R}^{n+1}$,

$$\psi_i(x) = \begin{cases} (\rho_i(x), \rho_i(x)\cdot\varphi_i^{-1}(x)) & , \quad x \in U_i \\ 0 & , \quad \text{else} \end{cases}$$

By Pasteing Lemma, ψ_i is cts.

Defn $\psi : M \to \mathbb{R}^{n\cdot N+N}$, $\psi(x) = (\ldots, \psi_i(x), \ldots)$.

Note, ψ is cts.

If $\psi(x) = \psi(y) \Rightarrow \psi_i(x) = \psi_i(y)\ \forall i$

$\exists\ i$ st $\rho_i(x) = 1$ since $\varphi_i(B(1))$ cover.

If $\rho_i(y) \neq 1 \Rightarrow \psi_i(y) \neq \psi_i(x)$

$\Rightarrow \varphi_i^{-1}(x) = \varphi_i^{-1}(y)$.

φ_i inj $\Rightarrow x = y$

So ψ is injective

$\Rightarrow \psi : M \to \psi(M)$ is cts bij from M = cpt to $\psi(M)$ = Haus

$\Rightarrow \psi$ is homeo onto $\psi(M)$

$\Rightarrow M$ = emb. manifold. $\square$

Paracompact Spaces

Defn: X = Space, $\{U_\alpha\}_\alpha$ = open cover of X.

① $\{U_\alpha\}$ is locally finite iff $\forall\ x \in X\ \exists$ open $x \in U$ st U meets only finitely many U_α's.

② A refinement of $\{U_\alpha\}_\alpha$ is an open cover $\{V_\beta\}_\beta$ of X st $\forall\ \beta\ \exists\ \alpha$ w/ $V_\beta \subseteq U_\alpha$

Defn: X = space is <u>paracompact</u> iff it is Haus and every open cover admits a locally finite refinement.

Warning: Some sources do not require paracompact spaces to be Hausdorff.

Ex: X = cpt, Haus $\Rightarrow$ X = paracpt.

Fact: Every metric space is paracpt

Lemma: $\mathbb{R}^n$ is paracompact

Proof: Spse $\bigcup_\alpha U_\alpha = \mathbb{R}^n$

Note $\mathbb{R}^n = \bigcup_n \overline{B_0(n)}$

Heine-Borel $\Rightarrow$ $\overline{B_0(n)}$ = cpt

$\Rightarrow \forall n, \exists\, U^n_{i_1}, \dots, U^n_{i_n}$ that cover $\overline{B_0(n)}$.

$V^n_{i_j} := U^n_{i_j} \cap (X \setminus \overline{B_0(n-1)})$; these cover $\overline{B_0(n)} \setminus \overline{B_0(n-1)}$

Note, $V^n_{i_j} \subseteq U^n_{i_j} \Rightarrow \bigcup \bigcup_j V_{i_j}$ are a refinement

$\bigcup_j V^n_{i_j}$ cover $\overline{B(n)} \setminus \overline{B(n-1)} \Rightarrow \bigcup_m \bigcup_j V^m_{i_j}$ cover X.

Spse $x \in \mathbb{R}^n \Rightarrow x \in B(n)$ for some n.

$B(n)$ only meets $\bigcup_{m=1}^n \bigcup_j V^m_{i_j}$ = finite # of opens.

$\Rightarrow \{V^m_{i_j}\}$ = locally finite. $\square$

Warning: Quotients of paracpt $\neq$ paracpt.

↳ For example, the fat point.

Warning: $X, Y = \text{paracpt} \not\Rightarrow X \times Y = \text{paracpt}$

$\hookrightarrow$ $\mathbb{R}$ w/ basis $\{[a,b)\} = \text{normal}$,

$(\mathbb{R}, \{[a,b)\}) \times (\mathbb{R}, \{[a,b)\}) \neq \text{normal} \neq \text{paracpt}.$

Thm: $X = \text{paracpt} \Rightarrow X = \text{normal}$

Warning: Normal $\not\Rightarrow$ paracompact

Prop: $A \subseteq X = \text{paracpt}$ w/ $A = \text{closed} \Rightarrow A = \text{paracpt}$

Warning: $A \subseteq X = \text{paracpt} \not\Rightarrow A = \text{paracpt}$