Lecture #8 - October 3rd, 2023

Defn: $X = T_1$ is <u>normal</u> if $\forall$ closed $A, B \subseteq X$ st $A \cap B = \emptyset$, there exist $\mathcal{U}, V \subseteq X$ open st $A \subseteq \mathcal{U}$, $B \subseteq V$, and $\mathcal{U} \cap V = \emptyset$.

Lemma: X is normal iff $\forall$ $C \subset X$ closed st $C \subset \mathcal{U}$ open, $\exists$ V= open st $C \subset V \subset \overline{V} \subset \mathcal{U}$.

Theorem: (Urysohn's Theorem) X = normal $\iff$ $X = T_1$ & $\forall$ $A, B \subset X$ disj and closed $\exists$ cts $f : X \to I$ st $f(a) = 0$ if $a \in A$
$f(b) = 1$ if $b \in B$

Defn: X is second countable if it has a countable basis.

Defn: A space is metrizable if it is homeo to a metric space

Theorem: (Urysohn's metrization theorem)
X = normal + second countable $\Rightarrow$ X = metrizable.

Proof: Step 1: For each $q \in \mathbb{Q} \cap [0,1]$, $\exists$ open $\mathcal{U}_q \subseteq X$ st
$p < q \Rightarrow \overline{\mathcal{U}_p} \subseteq \mathcal{U}_q$

↳ Proof: $\mathcal{U}_1 = X \setminus B$

By lemma, $\exists$ $\mathcal{U}_0 \subseteq X$ open st $A \subseteq \mathcal{U}_0 \subseteq \overline{\mathcal{U}_0} \subseteq \mathcal{U}_1$.

List $q_0 = 0$, $q_1 = 1$, q_2, q_3, ... #'s of $\mathbb{Q} \cap [0,1]$.

Write $Q_n = \{q_0, \dots, q_n\}$.

Spse we have constructed $\mathcal{U}_q$ for $q \in Q_n$ st if $p < q$ and $p, q \in Q_n \Rightarrow \overline{\mathcal{U}_p} \subseteq \mathcal{U}_q$.

Consider $s \in Q_{n+1} \setminus Q_n$.

Let $r \in Q_n$ be the largest # smaller than s

" $t \in Q$ " " smallest " larger " s

$\Rightarrow \overline{U_r} \subseteq U_t$.

By lemma, $\exists\, U_s$ = open st $\overline{U_r} \subseteq U_s \subseteq \overline{U_s} \subseteq U_t$.

$\Rightarrow$ Constructed U_q for $q \in Q_{n+1}$.

By induction, we have the desired subsets. $\square$

Step 2: For $q \in \mathbb{Q}_{\geq 0}$, define $U_q = X$ for $q > 1$.

Note $p < q \Rightarrow \overline{U_p} \subseteq U_q$

Step 3: Define $f: X \longrightarrow I$ by $f(x) = \inf\{q \mid x \in U_q\}$.

If $x \in A$, $f(x) = 0$ since $A \subseteq U_0$

If $x \in B$, $f(x) = 1$ since $x \in U_q$ for $q > 1$, but $x \notin U_1$.

Step 4: Show f is continuous

↳ Proof: Notice that

Ⓐ $x \in \overline{U_r} \Rightarrow f(x) \leq r$ $\quad (f(x) > r \Rightarrow x \notin \overline{U_r})$

Ⓑ $x \notin U_r \Rightarrow f(x) \geq r$ $\quad (f(x) < r \Rightarrow x \in U_r)$

NTS $\forall f(x) \in (a,b)$, $\exists\, x \in U$ = open st $f(U) \subseteq (a,b)$

Fix $a < p < f(x) < q < b$, $U := U_q \setminus \overline{U_p}$ = open

(i) $\left.\begin{array}{l}\text{By Ⓐ, } x \notin \overline{U_p} \\ \text{By Ⓑ, } x \in U_q\end{array}\right\} \Rightarrow x \in U$.

(ii) $\left.\begin{array}{l}\text{By Ⓐ, } f(U) \leq q \\ \text{By Ⓑ, } f(U) \geq p\end{array}\right\} \Rightarrow f(U) \subseteq (a,b)$ $\square$

Defn: $I^\omega = \{\underline{x} = (x_1, \dots, x_n, \dots) \mid x_i \in [0,1]\}$

$d: I^\omega \times I^\omega \longrightarrow \mathbb{R}_{\geq 0}$ by $d(\underline{x}, \underline{y}) = \sup_i |x_i - y_i|$ is a metric.

Proof: Only finitely many $x_i \neq 0$, $y_i \neq 0$ $\Rightarrow$ d is well-defined.

① By defn, d is symmetric.

② $d(\underline{x}, \underline{y}) = 0 \iff |x_i - y_i| = 0 \ \forall i \iff \underline{x} = \underline{y}$.

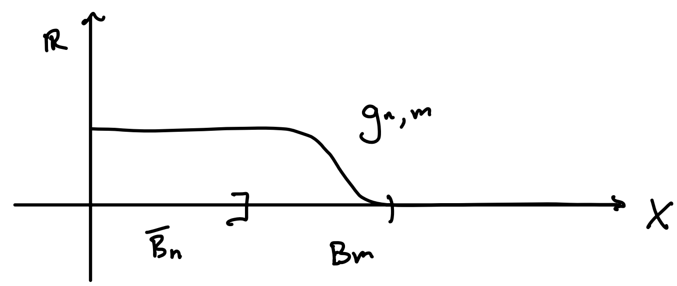
③ $$\begin{aligned} d(\underline{x}, \underline{z}) &= \sup_i |x_i - z_i| \\ &\leq \sup_i |x_i - y_i| + \sup_i |y_i - z_i| \\ &= d(\underline{x}, \underline{y}) + d(\underline{y}, \underline{z}) \end{aligned}$$ $\square$

Proof: Step 0: Construct a map $f : X \longrightarrow I^{\omega}$

Let $B_1, \ldots, B_n, \ldots$ be the countable basis.

Given $\overline{B_n} \subseteq B_m$, use Urysohn's lemma to obtain $g_{n,m} : X \rightarrow [0,1]$

st $g_{n,m}(\overline{B_n}) = 1$, $g_{n,m}(X \setminus B_m) = 0$.



Enumerate the $g_{n,m}$ as $g_i : X \rightarrow I$ and define $f_i = g_i / i$

Define $f : X \rightarrow I^{\omega}$ by $f(x) = (f_1(x), f_2(x), \ldots)$

(✱) $x \in B$, then Normal $\Rightarrow \{x\} \subseteq U \subseteq \overline{U} \subseteq B$.

$\Rightarrow \exists B'$ st $\{x\} \subseteq B' \subseteq \overline{B'} \subseteq B$.

Step 1: Show f is injective

$x \neq y \Rightarrow \exists$ basic open $x \in B_k$ st $y \in B_\ell$ w/ $B_k \cap B_\ell = \emptyset$.

By (✱), $\exists$ $x \in B_i \subseteq \overline{B_i} \subseteq B_k \Rightarrow g_{i,k}(x) = 1$, $g_{i,k}(y) = 0$

$\Rightarrow f(x) \neq f(y)$

Step 2: Show f is continuous.

Fix $x \in X$, $\varepsilon > 0$.

NTS $\exists\, x \in \mathcal{U} \subseteq X$ open st $y \in \mathcal{U} \Rightarrow d(f(x), f(y)) < \varepsilon$.

Fix N st $1/N < \varepsilon/2$.

f_n cts $\Rightarrow \exists\, \mathcal{U}_n \ni x$ open st $|f_n(x) - f_n(y)| \leq \varepsilon/2 \quad \forall y \in \mathcal{U}_n$.

Defn $\mathcal{U} = \mathcal{U}_1 \cap \dots \cap \mathcal{U}_N$

So if $y \in \mathcal{U}$, then

$$\begin{aligned} d(f(x), f(y)) &\leq d((f_1(x), \dots, f_N(x), 0, \dots), (f_1(y), \dots, f_N(y), 0, \dots)) \\ &\quad + d((0, \dots, 0, f_{N+1}(x), \dots), (0, \dots, 0, f_{N+1}(y), \dots)) \\ &\leq \frac{\varepsilon}{2} + \frac{1}{N+1} \\ &< \varepsilon \end{aligned}$$

Step 3: $\mathcal{U} \subseteq X$ open $\Rightarrow f(\mathcal{U}) \subseteq f(X)$ open in subspace top.

NTS $\forall z \in f(\mathcal{U})$, $\exists\, V$ open st $z \in V \cap f(X) \subseteq f(\mathcal{U})$

Fix $x \in \mathcal{U}$, N st $f_N(x) \neq 0$, $f_N(X \setminus B_m) = 0$, $f(x) = z$.

$\hookrightarrow$ By ($\star$), $g_{n,m}(\bar{B}_n) = 1$, $g_{n,m}(X \setminus B_m) \equiv 0$

w/ $x \in B_n \subseteq \bar{B}_n \subseteq B_m \subseteq \mathcal{U}$

Define $V = \{(x_1, \dots, x_n, \dots) \mid x_N \in (0, 1]\}$ = open

Spse $w \in V \cap f(X) \Rightarrow f(y) = w$ for some $y \in X$ w/ $f_N(y) > 0$.

$$\begin{aligned} &\Rightarrow y \notin X \setminus B_m \\ &\Rightarrow y \in B_m \\ &\Rightarrow y \in \mathcal{U} \end{aligned}$$

$\Rightarrow V \cap f(X) \subseteq f(\mathcal{U})$, as desired.

Step 4: Wrap-up

By above steps, we have a continuous bij. $f : X \longrightarrow f(X) \subseteq I^\omega$

Let $g : f(X) \to X$ = set theoretic inverse of f.

By 3, if $\mathcal{U} \subseteq X$ open, then $g^{-1}(\mathcal{U}) = f(\mathcal{U})$ = open $\Rightarrow g$ = cts.

$\Rightarrow X \cong f(X)$ w/ subspace top = metric top.

$\Rightarrow X \cong$ metric space. $\square$