Lecture #7 - September 28th, 2023

Defn: A top. space X is Hausdorff if for each $x, y \in X$ w/ $x \neq y$, $\exists$ opens U, V st $x \in U$, $y \in V$, and $U \cap V = \emptyset$.

Prop: X, Y = Haus

1. $x \in X \Rightarrow \{x\}$ = closed (so $X = T_1$)
2. $A \subseteq X$ subspace $\Rightarrow$ A Hausdorff
3. $X \times Y$ = Haus
4. $A \subseteq X$ cpt subspace $\Rightarrow$ A closed.
5. $x_n \in X$ st $x_n \to x$; $x_n \to y \Rightarrow x = y$.

Cor: $f: X \to Y$ cts + bij st X = cpt, Y = Haus $\Rightarrow f$ = homeo.

Proof: Let $g: Y \to X$ = set theoretic inverse of f.
NTS g is cts.
X cpt + $C \subseteq X$ closed $\Rightarrow C$ = cpt $\Rightarrow g^{-1}(C) = f(C)$ = cpt
Y Haus $\Rightarrow g^{-1}(C)$ = closed. $\square$

Normal Spaces

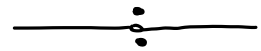
Defn: X is T_1 if $\{x\}$ = closed.

Defn: $X = T_1$ is normal if $\forall$ closed $A, B \subseteq X$ st $A \cap B = \emptyset$, there exist $U, V \subseteq X$ open st $A \subseteq U$, $B \subseteq V$, and $U \cap V = \emptyset$.

Rem:

① Normal $\Rightarrow$ Hausdorff $\Rightarrow$ T_1

② $T_1 \not\Rightarrow$ Hausdorff

$\hookrightarrow$ Fat point : $\mathbb{R}_1 \sqcup \mathbb{R}_2 / \sim$ = 

③ Hausdorff $\not\Rightarrow$ normal

$\hookrightarrow$ $\mathbb{R}$ w/ top. $\mathcal{O}$ gen. by the basis

$$\mathcal{B} = \{(a,b)\} \cup \{(a,b) \setminus \cup_n \{1/n\}\}$$

ⓐ $\mathcal{B}$ is a basis : covers $\mathbb{R}$, $B_1, B_2 \in \mathcal{B} \Rightarrow B_1 \cap B_2 \in \mathcal{B}$.

ⓑ $(\mathbb{R}, \mathcal{O})$ = Haus : $\mathcal{O}$ contains all the opens of std top. on $\mathbb{R}$

Since std top = Haus $\Rightarrow$ $\mathcal{O}$ is Haus.

ⓒ $(\mathbb{R}, \mathcal{O}) \neq$ normal: Consider $A = \{0\}$, $B = \cup_n \{1/n\}$ = closed

Spse $A \subseteq \mathcal{U}$ = open st $\mathcal{U} \cap B = \emptyset$ $\Rightarrow$ $(-\varepsilon, \varepsilon) \setminus \cup_n 1/n \subseteq \mathcal{U}$.

Spse $B \subseteq V$ = open $\Rightarrow$ $\frac{1}{N} \in V$ for $N >> 0$.

$\Rightarrow$ V contains $(\frac{1}{N} - \delta, \frac{1}{N} + \delta)$ for small $\delta > 0$

$\Rightarrow$ $V \cap \mathcal{U} \neq \emptyset$.

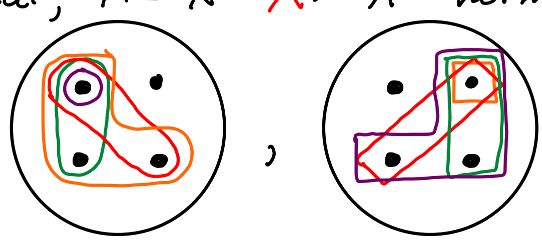
$\Rightarrow$ not normal

Warning:

① Quotient of a normal space need not be normal.

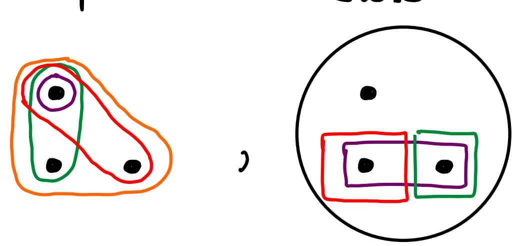
② X = normal, $A \subseteq X$ $\cancel{\Rightarrow}$ A = normal

Counter-examples that are T_1 are somewhat involved. We won't concern ourselves w/ them.

$\hookrightarrow$ X = 

$X \neq T_1$, but closed disj in X are separable by opens

Open closed

A = 

Can't separate □, □

③ $\mathbb{R}$ w/ basis $\{[a,b)\}$ = normal (exer)

But $\mathbb{R}^2$ w/ the product topology is not normal

Lemma: X = normal, $A \subseteq X$ closed $\Rightarrow$ A = normal

Proof: $C_0, C_1 \subseteq A$ closed st $C_0 \cap C_1 = \emptyset$.

$\Rightarrow C_0, C_1 \subseteq X$ closed st $C_0 \cap C_1 = \emptyset$

$\Rightarrow \exists$ opens in X, $\mathcal{U}_0 \supset C_0$, $\mathcal{U}_1 \supset C_1$ st $\mathcal{U}_0 \cap \mathcal{U}_1 = \emptyset$

$\Rightarrow A \cap \mathcal{U}_0$, $A \cap \mathcal{U}_1$ are desired separating opens. $\square$

Prop: X = cpt + Haus $\Rightarrow$ X = normal

Proof: Fix $A, B \subseteq X$ st A, B = closed and $A \cap B = \emptyset$.

Fix $x \in A$, $\forall y \in B$ $\exists$ $\mathcal{U}_y, V_y$ open st $x \in \mathcal{U}_y$, $y \in V_y$, $\mathcal{U}_y \cap V_y = \emptyset$

Since $B \subseteq X$ closed $\Rightarrow$ B = cpt

$\Rightarrow \exists$ $V_{y_1}, \dots, V_{y_n}$ st $V_x = \bigcup_i V_{y_i}$, $B \subseteq V_x$

Define $\mathcal{U}_x = \bigcap_i \mathcal{U}_{y_i}$ = open.

So $x \in \mathcal{U}_x$, $B \subseteq V_x$, and $\mathcal{U}_x \cap V_x = \emptyset$.

As w/ B, A is cpt. Also $A \subseteq \bigcup_x \mathcal{U}_x$

$\Rightarrow \exists$ $\mathcal{U}_{x_1}, \dots, \mathcal{U}_{x_m}$ st $A \subseteq \bigcup_i \mathcal{U}_{x_i}$.

Define $\mathcal{U} = \bigcup_{i=1}^m \mathcal{U}_{x_i}$, $V = \bigcap_{i=1}^m V_{x_i}$. So $A \subseteq \mathcal{U}$.

Since $B \subset V_{x_i}$ $\forall i$ $\Rightarrow$ $B \subseteq V$.

Finally, $\mathcal{U} \cap V = \emptyset$. So $\mathcal{U}, V$ give the desired separating nhoods $\square$

Prop: X = metric space $\Rightarrow$ X = normal.

Lemma: Given $A \subseteq X$ subset w/ X = metric space, the function

$$d(-, A) : X \longrightarrow \mathbb{R}, \quad d(x, A) = \inf_{a \in A} d(x, a)$$

is cts.

Proof: Fix $\varepsilon > 0$. Fix $x, y \in X$ st $d(x,y) < \varepsilon/2$

Fix $a \in A$ st $d(x,a) < d(x,A) + \delta$

$$d(y,A) \le d(y,a) \le d(y,x) + d(x,a) < \varepsilon/2 + \delta + d(x,A)$$

$\Rightarrow d(y,A) \le \varepsilon/2 + d(x,A) \Rightarrow d(y,A) - d(x,A) \le \varepsilon/2$

Switch roles of x,y $\Rightarrow d(x,A) - d(y,A) \le \varepsilon/2$

$\Rightarrow |d(x,A) - d(y,A)| < \varepsilon$

ε-δ defn of cts gives result w/ $\delta = \varepsilon/2$ $\square$

Proof: (Proof of above proposition)

Let $A, B \subseteq X$ be disjoint + closed.

$\mathcal{U} = \{x \mid d(x,A) - d(x,B) < 0\}$

$V = \{x \mid d(x,A) - d(x,B) > 0\}$

$d(-,A) - d(-,B) = $ cts $\Rightarrow \mathcal{U}, V = $ open + disjoint

NTS $A \subseteq \mathcal{U}$, $B \subseteq V$.

Spse $a \in A$ st $d(a,A) - d(a,B) \le 0$

If $d(a,B) = 0 \Rightarrow a = $ lim pt of $B \Rightarrow a \in B = $ closed $\Rightarrow A \cap B \neq \phi$

$\Rightarrow d(a,A) - d(a,B) \le 0$

$\Rightarrow A \subseteq \mathcal{U}$

Similarly, $B \subseteq V$. $\square$

Lemma: X is normal iff $\forall\, C \subset X$ closed st $C \subset \mathcal{U}$ open, $\exists\, V = $ open st $C \subset V \subset \overline{V} \subset \mathcal{U}$.

Proof: ($\Rightarrow$) $A = X \setminus \mathcal{U} = $ closed.

$\Rightarrow \exists\, V, W$ open st $C \subset V$, $A \subset W$, $V \cap W = \phi$.

$V \subseteq X \setminus W = $ closed $\Rightarrow \overline{V} \subseteq X \setminus W \subseteq X \setminus \mathcal{U} \Rightarrow \overline{V} \subsetneq \mathcal{U}$

$\Rightarrow C \subset V \subset \overline{V} \subset \mathcal{U}$.

($\Leftarrow$) $A, B =$ closed w/ $A \cap B = \emptyset$

$A \subseteq X \setminus B =$ open

$\Rightarrow \exists\ V =$ open w/ $A \subseteq V \subseteq \bar{V} \subseteq X \setminus B$

$\Rightarrow B \subseteq X \setminus \bar{V} =$ open

Also $V \cap (X \setminus \bar{V}) = \emptyset$ $\square$

Theorem: (Urysohn's Theorem) $X =$ normal, $A, B \subset X$ disj + closed

$\Rightarrow \exists$ cts $f : X \to I$ st $f(a) = 0$ if $a \in A$
$f(b) = 1$ if $b \in B$

Theorem: (Tietz extension theorem) If $A \subseteq X$ closed and X normal, then every cts $f : A \to \mathbb{R}$ admits a cts extension $\tilde{f} : X \to \mathbb{R}$

$\hookrightarrow \tilde{f}|_A = f$.

Cor: $X =$ metric space. $X =$ compact iff each $f : X \to \mathbb{R}$ is cts.

Proof: Since $X =$ metric $\Rightarrow$ cpt iff seq. cpt.

Spse $x_n \in X$ st x_n has no convergent subsequence. $\rightsquigarrow$ Assume x_n are distinct.

Spse $x =$ limit point of $\bigcup_n x_n$

$x_n \not\to x$, so $B_x(\varepsilon) \cap \{x_n\}$ must be finite for some $\varepsilon > 0$.

So for ε suff. small, $B_x(\varepsilon) \cap \{x\} = \emptyset$ when $x \notin \bigcup_n x_n$

$\Rightarrow \bigcup_n x_n =$ closed and for distinct x_n $\exists\ B_{x_n}(\varepsilon_n)$ that misses all other x_i

$f : \bigcup_n x_n \to \mathbb{R}$, $f(x_n) = n$ $\rightsquigarrow$ f is cts by ε-δ argument.

By Tietz, f extends to $\tilde{f} : X \to \mathbb{R}$ st $\tilde{f} \neq$ bounded $\Rightarrow \Leftarrow$ $\square$