Lecture #6 - September 26^{th}, 2023

Defn: X = space is compact if each collection $\mathcal{U}_\alpha$ of opens st $X = \bigcup_\alpha \mathcal{U}_\alpha$, $\exists\ \mathcal{U}_{\alpha_1}, \dots, \mathcal{U}_{\alpha_n}$ st $X = \bigcup_i \mathcal{U}_{\alpha_i}$.

↪ Every open cover admits finite subcover.

Defn: X = sequentially compact iff $\forall\ x_n$, $\exists$ subseq $x_{n_i} \longrightarrow x$.

Theorem: X = cpt + 1^{st} countable $\Rightarrow X$ = seq. compact ($\Leftarrow$ if X = metric space)

Proof: Spse the x_n do not have a convergent subsequence.

$\exists\ V_1 \supset \dots \supset V_n, \dots$ opens st $x \in \mathcal{U} \Rightarrow \exists\ i \gg 0$ st $V_i \subseteq \mathcal{U}$.

only finitely many x_n lie in some V_i for i suff. larger

↪ If not, then $\exists$ a subseq that converges to x.

$\Rightarrow \forall\ x \in X, \exists\ \mathcal{U}_x \ni x$ open st $\mathcal{U}_x \cap \bigcup_n x_n$ is finite.

$X = \bigcup_x \mathcal{U}_x \Rightarrow \exists\ \mathcal{U}_{x_1}, \dots, \mathcal{U}_{x_k}$ st $X = \bigcup_k \mathcal{U}_{x_k}$.

Since each $\mathcal{U}_{x_i}$ contains only finitely many x_n in the sequence

$\Rightarrow \exists$ finite # of x_n's $\Rightarrow \Leftarrow$. $\square$

Thm: (Heine-Borel) Any subspace $X \subset \mathbb{R}^n$ is cpt iff it is closed and bounded

Lemma: X = cpt, $C_\alpha \supset C_{\alpha+1} \supset C_{\alpha+2} \supset \cdots$ w/ C_α = closed + non-empty

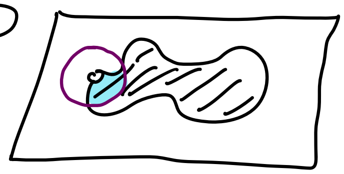
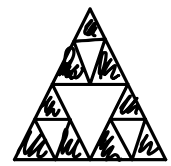
$\Rightarrow \bigcap_\alpha C_\alpha \neq \emptyset$.

Proof: Spse $\bigcap_\alpha C_\alpha = \phi$, then $U_\alpha = X \setminus C_\alpha$ give $X = \bigcup_\alpha U_\alpha$

$\Rightarrow \exists\, U_1, \dots, U_n$ st $X = \bigcup_i U_i$

Note $U_\alpha \subset U_{\alpha+1}$. So for some $N \gg 0$, $U_N = X$.

$\Rightarrow C_N = \phi \Rightarrow\Leftarrow$ $\square$

Ex: ①  $\neq$ cpt ②  $\rightsquigarrow \bigcap_\alpha C_\alpha$ = complicated

Prop: X = cpt, $C \subseteq X$ closed $\Rightarrow$ C is cpt.

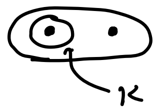
Proof: Spse $C = \bigcup_\alpha C \cap U_\alpha$

$\Rightarrow X = \bigcup_\alpha U_\alpha \cup (X \setminus C)$

$\Rightarrow \exists\, U_1, \dots, U_n$ st $X = \bigcup_i U_i \cup (X \setminus C)$

$\Rightarrow C = \bigcup_i C \cap U_i$ $\square$

Warning: X = cpt, $K \subset X$ cpt $\not\Rightarrow$ K closed

$\hookrightarrow X =$ 

, K is <u>not</u> closed

Theorem: $[a,b]$ is compact

Proof: Spse $[a,b] = \bigcup_\alpha V_\alpha$

Let $S = \{ x \in [a,b] \mid [a,x]$ is covered by finitely many $U_\alpha \}$.

Note $S \neq \phi$ and S is bounded by 1.

$\Rightarrow \exists$ lub L of S.

Claim: $L \in S$.

$\hookrightarrow$ Spse $L \in \mathcal{U} \Rightarrow \exists\, \varepsilon > 0$ st $(L-\varepsilon, L] \subset \mathcal{U}$

$\exists$ finite $\mathcal{U}_1, \dots, \mathcal{U}_n$ that cover $[a, L - \varepsilon/2]$.

$\Rightarrow \mathcal{U}, \mathcal{U}_1, \dots, \mathcal{U}_n$ cover $[a, L]$

Claim: $L = 1$

$\hookrightarrow$ Spse $L \neq 1$, let $L \in \mathcal{U} \Rightarrow \exists\, \varepsilon > 0$ st $[L, L+\varepsilon) \subset \mathcal{U}$.

$\exists$ finite $\mathcal{U}_1, \dots, \mathcal{U}_n$ that cover $[0, L]$.

$\Rightarrow \mathcal{U}, \mathcal{U}_1, \dots, \mathcal{U}_n$ cover $[a, L+\varepsilon/2]$

$\Rightarrow L + \varepsilon/2 \in S \Rightarrow L \neq$ lub $\Rightarrow \Leftarrow$ $\square$

Theorem: If $X, Y = $ cpt, then $X \times Y = $ cpt.

Proof: ① The map $pr_X : X \times Y \longrightarrow X$ is open,

i.e., $pr_X(O) = $ open $\forall\, O \subset X \times Y$ open.

$\hookrightarrow$ $O = \bigcup_\alpha U_\alpha \times V_\alpha$ for $U_\alpha \subseteq X$, $V_\alpha \subseteq Y$ opens

$pr_X(O) = \bigcup_\alpha U_\alpha = $ open.

② Spse $X \times Y = \bigcup_\alpha O_\alpha$, WLOG assume $O_\alpha = U_\alpha \times V_\alpha$.

Since $Y = $ cpt $\Rightarrow \exists\, U_{x,i} \times V_{x,i}$ that cover $\{x\} \times Y$

Set $U_x = \bigcap_i U_{x,i}$. $\hookrightarrow$ w/ $U_{x,i} = U_\alpha$, $V_{x,i} = V_\alpha$ for some α's.

$\Rightarrow U_x \times V_{x,i}$ cover $U_x \times Y$ and are contained in $U_{x,i} \times V_{x,i}$.

As x varies, $\exists\, U_{x_1}, \dots, U_{x_m}$ that cover X.

$\Rightarrow U_{x_i,j} \times V_{x_i,j}$ cover X $\square$

Proof: $(\Leftarrow)$: X bounded $\Rightarrow X \subseteq [-L, L]^n$ for some $L \gg 0$.

$[-L, L]^n$ is compact and X is closed $\Rightarrow X = $ cpt

$\hookrightarrow$ (since $[-L, L]^n \subset \mathbb{R}^n$ closed, HW 2)

$(\Rightarrow)$: $X = \bigcup_{r\in\mathbb{R}_{>0}} X\cap B_r(0) \Rightarrow \exists\, r$ st $X = X\cap B_r(0) \Rightarrow X$ bdd.

Spse $x\in\mathbb{R}^n$ is a limit pt of X.

$\Rightarrow \forall r>0 \quad B_r(x)\cap X \neq \emptyset \Rightarrow C_n = \overline{B_{\frac{1}{n}}(x)} \cap X \neq \emptyset$.

So $C_n \supset C_{n+1} \supset \cdots$ = nested seq. of closed, non-empty in X.

$\Rightarrow \emptyset \neq \bigcap_n C_n = X \cap \bigcap_n \overline{B_{1/n}(x)} = X\cap\{x\} \Rightarrow x\in X$.

$\Rightarrow X$ has all its limit pts $\Rightarrow X$ = closed. $\square$

Hausdorff Spaces

Defn: A top. space X is <u>Hausdorff</u> if for each $x,y\in X$ w/ $x\neq y$, $\exists$ opens U,V st $x\in U$, $y\in V$, and $U\cap V=\emptyset$.

Ex:

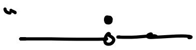
① X = triv. top. $\Rightarrow$ not Hausdorff

② X = metric space $\Rightarrow$ Hausdorff

$\hookrightarrow$ $x,y\in X$, $\ell = d(x,y)$, $B_x(\ell/2)\cap B_y(\ell/2)=\emptyset$.

③ $X=\mathbb{R}$ w/ opens $= \mathbb{R}\setminus\{x_1,\dots,x_n\}$.

Any two non-empty opens will have non-empty intersection.

④ Fat point, $\mathbb{R}_1\sqcup\mathbb{R}_2/\sim$ = "  " $\neq$ Hausdorff.

Lemma: X = Haus iff $\Delta = \{(x,x)\in X\times X\}\subseteq X\times X$ is closed.

Proof: $(\Rightarrow)$: X = Haus $\Rightarrow \forall x\neq y$, $\exists\, U_{xy}, V_{xy}$ st $x\in U_{xy}$, $y\in V_{xy}$, $U_{xy}\cap V_{xy}=\emptyset$.

$W_{xy} = U_{xy}\times V_{xy} \subset X\times X$.

$W_{xy}\cap\Delta=\emptyset$ and $\bigcup_{xy} W_{xy} = X\times X\setminus\Delta \Rightarrow \Delta$ = closed

$(\Leftarrow)$: Δ = closed $\Rightarrow \forall x\neq y$, $\exists$ basic open $U\times V\subseteq X\times X\setminus\Delta$ w/ $(x,y)\in U\times V$.

Note, $U\cap V=\emptyset$ since $U\times V\cap\Delta=\emptyset$. Also $x\in U$, $y\in V$. $\Rightarrow X$ Haus $\square$

Lemma: $X = \text{Haus}$, $x_n \in X$ st $x_n \to x$ & $x_n \to y \Rightarrow x = y$.

Proof: $\exists$ opens $x \in \mathcal{U}_x$, $y \in \mathcal{U}_y$ st $\mathcal{U}_x \cap \mathcal{U}_y = \emptyset$.

$x_n \to x \Rightarrow \exists N$ st $x_n \in \mathcal{U}_x \; \forall n > N$

$\Rightarrow x_n \notin \mathcal{U}_y$ for $n > N$

$\Rightarrow x_n \not\to y$. $\square$

Prop: $X, Y = \text{Haus}$

① $x \in X \Rightarrow \{x\} = \text{closed}$ (So $X = T_1$)

② $A \subseteq X$ subspace $\Rightarrow A$ Hausdorff

③ $X \times Y = \text{Haus}$

Warning: Quotient of Hausdorff space may not be Hausdorff.

Proof: ① $\forall y \neq x$, $\exists \mathcal{U}_y = \text{open}$ st $x \notin \mathcal{U}_y$

$\Rightarrow x = \bigcap_y (X \setminus \mathcal{U}_y) = \text{closed}$

② $x, y \in A$ w/ $x \neq y \Rightarrow \exists \mathcal{U}_x, \mathcal{U}_y \subseteq X$ open w/ $\mathcal{U}_x \cap \mathcal{U}_y = \emptyset$

$\Rightarrow (A \cap \mathcal{U}_x) \cap (A \cap \mathcal{U}_y) = \emptyset$

$\Rightarrow A$ Haus.

③ $(x_0, y_0) \in X \times Y$, $(x_1, y_1) \in X \times Y$ st $(x_0, y_0) \neq (x_1, y_1)$

Spse $x_0 \neq x_1 \Rightarrow \exists \mathcal{U}_0, \mathcal{U}_1 \subset X$ open w/ $x_i \in \mathcal{U}_i$, $\mathcal{U}_0 \cap \mathcal{U}_1 = \emptyset$.

$\Rightarrow \mathcal{U}_0 \times Y \cap \mathcal{U}_1 \times Y = (\mathcal{U}_0 \cap \mathcal{U}_1) \times Y = \emptyset$.

$\Rightarrow X \times Y = \text{Haus}$. $\square$

Lemma: X = Haus, $A \subseteq X$ cpt subspace $\Rightarrow$ A closed.

Proof: Fix $x \in X \setminus A$.

$\forall a \in A$, $\exists\ \mathcal{U}_a, V_a$ st $\mathcal{U}_a \cap V_a = \emptyset$.

$A = \bigcup_a A \cap \mathcal{U}_a \Rightarrow \exists\ \mathcal{U}_{a_1}, \dots, \mathcal{U}_{a_n}$ st $A = \bigcup_{i=1}^n A \cap \mathcal{U}_{a_i}$.

$\Rightarrow$ $V = \bigcap_{i=1}^n V_{a_i}$ satisfies $x \in V$, $A \cap V = \emptyset$.

$\Rightarrow$ $x \neq$ limit pt of A

$\Rightarrow$ A contains all its lim. pts

$\Rightarrow$ A = closed. $\square$