Lemma: TFAE

1. \( X = \text{conn.} \)
2. \( X \neq U_0 \cup U_i, \) st \( U_i \subseteq X \) is open, \( U_i \neq \emptyset. \)
3. \( X \neq C_0 \cup C_i, \) st \( C_i \subseteq X \) closed, \( C_i \neq \emptyset. \)

Definition: \( X \) is \textit{path-connected} if \( \forall a, b \in X, \exists \text{ path } \gamma: I \to X \text{ s.t. } \gamma(0) = a, \gamma(1) = b. \)

Examples:
1. \( \mathbb{R}^n, S^n = (\text{path}) - \text{conn.} \)
2. \( [0,1] = \text{conn.} \)

Lemma: Path-conn \( \implies \) connected.

Proposition: \( f: X \to Y \) cts + surj.

1. \( X \text{ conn } \implies Y \text{ conn } \)
2. \( X \text{ path-conn } \implies Y \text{ path-conn.} \)

Proof:
1. Spec \( Y = U_0 \cup U_i, \) w/ \( U_i = \text{open + non-empty} \)
\( \implies f^{-1}(U_i) = \text{open + non-empty} \) (since \( f \) surj)
\( \implies X = f^{-1}(Y) = f^{-1}(U_0) \cup f^{-1}(U_i) \)
\( \implies X \text{ not conn } \iff = \)

3. Spec \( a, b \in Y, \exists \tilde{a}, \tilde{b} \in X \text{ s.t. } f(\tilde{a}) = a, f(\tilde{b}) = b. \)
Since \( X = \text{path-conn } \implies \exists \text{ path } \gamma: I \to X \text{ w/ } \gamma(0) = \tilde{a}, \gamma(1) = \tilde{b}. \)
\( f \circ \gamma \) cts + \( f \circ \gamma(0) = a, f \circ \gamma(1) = b \)
\( \implies Y \text{ path-conn.} \)
Cor: Quotient of (path)-conn space is (path)-conn.

Cor: \(X \subseteq Y \implies X\) (path)-conn \iff \(Y\) (path)-conn.

Exercise: \(X, Y\) (path)-conn \iff \(X \times Y\) is (path)-conn.

Warning: \(A \subseteq X\) subspace \(\implies X\) (path)-conn \(\not\implies A\) (path)-conn.

Proposition: \(S^1, [a, b], (a, b), [a, b]\) are pairwise non-homeo.

Proof: \(|f : X \to Y\) homeo, \(x \in X \implies f|_{X \setminus x} : X \setminus x \to Y \setminus f(x)\) homeo.
So \(X \equiv Y \implies X \setminus x \equiv Y \setminus f(x)\) \(\forall x \in X\). \(\square\)

Topologist’s Sign Curve

Notn: \(X = \{(x, y) \in \mathbb{R}^2 \mid x = 0, 0 \leq y \leq 1\} \cup \{(x, \sin(\frac{1}{x})) \in \mathbb{R}^2 \mid x > 0\} = A \cup B\)

Lemma: \(A \subseteq X\) subspace w/ \(A\) conn \(\implies \overline{A}\) conn.

Proof: Supp \(\overline{A} = C_0 \cup C_1\) w/ \(C_i\) = closed + non-empty
\(\implies C'_i = A \cap C_i\), closed in A.
If \(C'_i = \emptyset\), then \(A \subseteq C_i \implies \overline{A} \subseteq C_i \implies C_0 = \emptyset \implies \square\)

Lemma: \(X = \text{conn}\).

Proof: \(B = \text{path-conn} \text{ and } \overline{B} = A \cup B = X \implies X = \text{conn}\). \(\square\)
Lemma: \( X \neq \text{path-conn.} \)

Proof: \( \gamma : I \to X \) s.t. \( \gamma(0) = (0,1) \in A \)
\( \gamma^{-1}(A) = \text{closed + non-empty} \)

Claim: \( \gamma^{-1}(A) = \text{open} \)

\[ \implies \gamma^{-1}(A) = I \implies \gamma(I) \subseteq A \ \forall \gamma = (0,1) \text{ not conn to any } b \in B. \]

Proof of claim: Fix \( t \in \gamma^{-1}(A) \), \( \exists t \in (a,b) \) s.t. \( \gamma((a,b)) \subseteq B \gamma(\epsilon) \cap X \)

\( B \gamma(\epsilon) \cap X \) has infty many path-comp one of which is completely contained in \( A \).
\( \gamma(t) \in A \implies \gamma((a,b)) \subseteq A \implies (a,b) \in \gamma^{-1}(A) = \text{open} \)

\[ \text{since image must be path-connected.} \]

Components

Lemma: \( A, B \subseteq X \) (path)-conn as subspaces and \( A \cap B \neq \emptyset \), then \( A \cup B \) is (path)-conn

Proof: (path-conn): Fix \( z \in A \cap B \), \( x \in A \), \( y \in B \)

\[ \exists \text{ path } \gamma_A : I \to A \text{ from } x \to z \]
\[ \gamma_B : I \to B \text{ from } z \to y. \]
\[ \gamma : I \to A \cup B, \gamma(t) = \begin{cases} \gamma_A(2t), & 0 \leq t \leq \frac{1}{2} \\ \gamma_B(2t-1), & \frac{1}{2} \leq t \leq 1 \end{cases} \]

\( \gamma \) is cts by the Paste-ing lemma.

\( \gamma \) conn \( x, y. \)

Conn: Spec \( A \cup B = U_0 \cup U_1 \) w/ \( U_i = \text{open + non-empty} \)
\( \Rightarrow V : = A \cap U_i = \text{open + closed} \Rightarrow \text{WLOG } V_1 = A \), \( V_2 = \emptyset \)
if \( V_0 = \emptyset \) \( \Rightarrow U_0 \notin B \setminus A \cap B \)

\[ \Rightarrow B \setminus U_0 = B \cap U_i \text{ open + non-empty} \]
\[ \Rightarrow B = U_1 \cup (B \cap U_1) \Rightarrow B \text{ not conn.} \]
Defn: __F equiv. rel. on X via__

- $x \sim y$ iff $x, y \in A \subseteq X$ w/ A conn.
- $x \equiv y$ iff $x, y$ are conn. by path.

$\Rightarrow$ Equiv. rel. by above lemma!

Lemma: __$x \equiv y \Rightarrow x \sim y$__
- Every component = union of some path-components

Proof: __$\gamma : I \rightarrow X$ conn $x$ and $y$.__

$\gamma(I) \subseteq X$ is a conn subspace that contains $x, y$.

Notn: __$C(x) = \{ y \in X | y \sim x \}$ = conn component of $x$__

$P(x) = \{ y \in X | y \equiv x \}$ = path-component of $x$.

- $x \sim y \Rightarrow C(x) = C(y)$
- $x \equiv y \Rightarrow P(x) = P(y)$

Remark: __$X = \bigsqcup_{c(x)} C(x) , \quad X = \bigsqcup_{c(x)} P(x)$__

Exer: __$\Phi : X \rightarrow Y$ cts $\Rightarrow \begin{cases} \Phi(C(x)) \subseteq C(\Phi(x)) \\ \Phi(P(x)) \subseteq P(\Phi(x)) \end{cases}$__

Cor: __$A \subseteq X$ (path)-conn $\Rightarrow A \subseteq C(x)$ (resp. $P(x)$) for $x \in A$.__

Compactness

Defn: __$X =$ space is compact if each collection $\mathcal{U}_a$ of opens st $X = \bigcup_{a \in A} \mathcal{U}_a$ , $\mathcal{U}_1, \ldots, \mathcal{U}_n \text{ st } X = \bigcup_{i = 1}^n \mathcal{U}_i$.__

$\Rightarrow$ Every open cover admits finite subcover.
Ex: ________
1. $\mathbb{R}$ is not compact
2. $(0,1)$ is not compact
3. $\mathbb{Q}$ is compact
4. $\mathbb{Q} \cap [0,1)$ is not compact
5. __________ \\

Defn: ________ $X \subseteq \mathbb{R}^n$ is bounded iff $\exists L$ st $|x| \leq L$ $\forall x \in X$.

Thm: ________ (Heine-Borel) Any subspace $X \subseteq \mathbb{R}^n$ is cpt iff it is closed and bounded

Proof: next time.

Lemma: ________ $f: X \to Y$ cts , $X$ = cpt $\Rightarrow f(X)$ = cpt

Proof: Fix a cover $f(X) = \bigcup \alpha f(x) \cap U_\alpha$ , $U_\alpha$ = open in $Y$.
$U_\alpha f^{-1}(U_\alpha)$ = open cover for $X$.
$\Rightarrow \exists U_1, \ldots, U_n$ s.t. $f^{-1}(U_1) \cup \cdots \cup f^{-1}(U_n) = X$.
$\Rightarrow f(X) = f(U; f^{-1}(U_1)) = U; f(x) \cup U_1$.
$\Rightarrow$ Admits finite subcover.

Ex: ________ $X$ = cpt $\iff$ any cts for $f: X \to \mathbb{R}$ is bounded

Cor: ________ $X \cong Y$ $\Rightarrow$ $X$ cpt iff $Y$ cpt.
Def: $X$ is sequentially compact iff $\forall x_n, \exists$ subseq $x_n \rightarrow x$.

Theorem: $X$ = cpt $\cap$ 1st countable $\Rightarrow X$ = seq. compact ($\Leftarrow$ if $X$ = metric space)

Proof: Suppose the $x_n$ do not have a convergent subsequence.

$\exists U_1, U_2, ...$ open s.t. $x \in U \Rightarrow \exists i > 0$ s.t. $V_i \subseteq U$.

Only finitely many $x_n$ lie in some $V_i$ for $i$ suff. larger.

$\Rightarrow$ If not, then $\exists$ a subseq that converges to $x$.

$\Rightarrow \forall x \in X, \exists x \in U$ open s.t. $U \cap U \cap ...$ is finite.

$X = \bigcup x U_x \Rightarrow \exists x, ..., U_x \cap x \cap \cap \sec X = \bigcup x U_x$.

Since each $U_x$ contains only finitely many $x_n$ in the sequence

$\Rightarrow \exists$ finite # of $x_n$'s $\Rightarrow \Leftarrow$. □