Lecture #5 - September 21st, 2023

Lemma: TFAE

(i) X = conn.

(ii) $X \neq U_0 \sqcup U_1$ st $U_i \subset X$ is open, $U_i \neq \emptyset$.

(iii) $X \neq C_0 \sqcup C_1$ " $C_i \subset X$ " closed, $C_i \neq \emptyset$.

Defn: X = space is path-connected if $\forall a, b \in X$, $\exists$ path $\gamma : I \to X$ st $\gamma(0) = a$, $\gamma(1) = b$.

Ex: (1) $\mathbb{R}^n, S^n$ = (path) conn.

(2) $[0,1]$ = conn.

Lemma: Path-conn $\Rightarrow$ connected.

Proposition: $f : X \to Y$ cts + surj

(i) X conn $\Rightarrow$ Y conn

(ii) X path-conn $\Rightarrow$ Y path-conn.

Proof: (i) Spse $Y = U_0 \sqcup U_1$ w/ U_i = open + non-empty

$\Rightarrow f^{-1}(U_i)$ = open + non-empty (since f surj)

$\Rightarrow X = f^{-1}(Y) = f^{-1}(U_0) \sqcup f^{-1}(U_1)$

$\Rightarrow X$ not conn $\Rightarrow\Leftarrow$

(ii) Spse $a, b \in Y$, $\exists$ $\tilde{a}, \tilde{b} \in X$ st $f(\tilde{a}) = a$, $f(\tilde{b}) = b$.

Since X = path-conn $\Rightarrow \exists$ path $\gamma : I \to X$ w/ $\gamma(0) = \tilde{a}$, $\gamma(1) = \tilde{b}$.

$f \circ \gamma$ cts + $f \circ \gamma(0) = a$, $f \circ \gamma(1) = b$

$\Rightarrow Y$ = path-conn. $\square$

Cor: Quotient of (path)-conn space is (path)-conn.

Cor $X \cong Y \Rightarrow X$ (path)-conn iff Y (path)-conn.

Exercise: X, Y (path)-conn $\Leftrightarrow X \times Y$ is (path)-conn.

Warning: $A \subseteq X$ subspace + $X =$ (path)-conn $\not\Rightarrow A =$ (path)-conn.

Proposition: S^1, $[a,b)$, (a,b), $[a,b]$ are pairwise non-homeo.

Proof: $f : X \to Y$ homeo, $x \in X \Rightarrow f|_{X \setminus x} : X \setminus x \to Y \setminus f(x)$ homeo

So $X \cong Y \Rightarrow X \setminus x \cong Y \setminus f(x)$ $\forall x \in X$. $\square$

Topologist's Sign Curve

Notn: $X = \{(x,y) \in \mathbb{R}^2 \mid x = 0, 0 \le y \le 1\} \sqcup \{(x, \sin(1/x)) \in \mathbb{R}^2 \mid x > 0\} = A \sqcup B$

Lemma: $A \subseteq X$ subspace w/ A conn $\Rightarrow \bar{A}$ conn.

Proof: Spse $\bar{A} = C_0 \cup C_1$ w/ $C_i =$ closed + non-empty

$\Rightarrow C_i' = A \cap C_i =$ closed in A.

If $C_0' = \phi$, then $A \subseteq C_1 \Rightarrow \bar{A} \subseteq C_1 \Rightarrow C_0 = \phi \Rightarrow\Leftarrow$ $\square$

Lemma: $X =$ conn.

Proof: $B =$ path-conn and $\overline{B} = A \cup B = X \Rightarrow X =$ conn. $\square$

Lemma: $X \neq$ path-conn.

Proof: $\gamma : I \to X$ st $\gamma(0) = (0,1) \in A$

$\gamma^{-1}(A)$ = closed + non-empty

Claim: $\gamma^{-1}(A)$ = open

since I = conn and $\gamma^{-1}(A)$ = open + closed

$\Rightarrow \gamma^{-1}(A) = I \Rightarrow \gamma(I) \subseteq A \ \forall \gamma \Rightarrow (0,1)$ not conn to any $b \in B$.

Pf of claim: Fix $t \in \gamma^{-1}(A)$, $\exists\ t \in (a,b)$ st $\gamma((a,b)) \subseteq B_{\gamma(t)}(\varepsilon) \cap X$

$B_{\gamma(t)}(\varepsilon) \cap X$ has infty many path-comp one of which is completely contained in A.

$\gamma(t) \in A \Rightarrow \gamma((a,b)) \subseteq A \Rightarrow (a,b) \subseteq \gamma^{-1}(A) \Rightarrow$ open $\square$

since image must be path-connected

Components

Lemma: $A, B \subseteq X$ (path)-conn as subspaces and $A \cap B \neq \emptyset$, then $A \cup B$ is (path)-conn

Proof: Path-conn.: Fix $z \in A \cap B$, $x \in A$, $y \in B$

$\exists$ path $\gamma_A : I \to A$ from x to z

" " $\gamma_B : I \to B$ " z " y.

$$\gamma : I \to A \cup B, \quad \gamma(t) = \begin{cases} \gamma_A(2 \cdot t), & 0 \le t \le 1/2 \\ \gamma_B(2t - 1), & 1/2 \le t \le 2 \end{cases}$$

γ is cts by the Pasteing lemma.

γ conn x, y. $\square$

Conn: Spse $A \cup B = \mathcal{U}_0 \sqcup \mathcal{U}_1$ w/ $\mathcal{U}_i$ = open + non-empty

$\Rightarrow V_i = A \cap \mathcal{U}_i$ = open + closed $\Rightarrow$ WLOG $V_1 = A$, $V_0 = \emptyset$

if $V_0 = \emptyset \Rightarrow \mathcal{U}_0 \subsetneq B \setminus A \cap B$

$\Rightarrow B \setminus \mathcal{U}_0 = B \cap \mathcal{U}_1$ = open + non-empty

$\Rightarrow B = \mathcal{U}_0 \sqcup (B \cap \mathcal{U}_1) \Rightarrow B$ not conn.

Defn: $\exists$ equiv. rel. on X via

① $x \sim y$ iff $x, y \in A \subseteq X$ w/ A conn.

② $x \approx y$ iff x, y are conn. by path.

↳ Equiv. rel. by above lemma!

Lemma:
- $x \approx y \Rightarrow x \sim y$
- Every component = union of some path-components

Proof: $\gamma : I \to X$ conn x and y.

$\gamma(I) \subseteq X$ is a conn subspace that contains x, y.

Notn: $C(x) = \{ y \in X \mid y \sim x \}$ = conn component of x

$P(x) = \{ y \in X \mid y \approx x \}$ = path-component of x.

↳ $x \sim y \Rightarrow C(x) = C(y)$

$x \approx y \Rightarrow P(x) = P(y)$

Remark: $X = \bigsqcup_{C(x)} C(x)$, $X = \bigsqcup_{P(x)} P(x)$

Exer: $f : X \to Y$ cts $\Rightarrow \begin{cases} f(C(x)) \subseteq C(f(x)) \\ f(P(x)) \subseteq P(f(x)) \end{cases}$

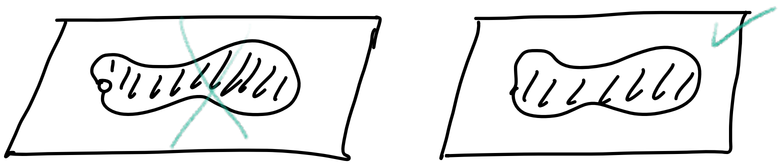
Cor: $A \subseteq X$ (path)-conn $\Rightarrow A \subseteq C(x)$ (resp. $P(x)$) for $x \in A$.

Compactness

Defn: X = space is compact if each collection $\mathcal{U}_\alpha$ of opens st $X = \bigcup_\alpha \mathcal{U}_\alpha$, $\exists\ \mathcal{U}_{\alpha_1}, \ldots, \mathcal{U}_{\alpha_n}$ st $X = \bigcup_i \mathcal{U}_{\alpha_i}$.

↳ Every open cover admits finite subcover.

Ex:

1. $\mathbb{R}$ is not compact
2. $(0,1)$ is not compact
3. $\mathbb{Q} \neq$ compact
4. $\mathbb{Q} \cap [0,1] \neq$ compact
5. 

Defn: $X \subseteq \mathbb{R}^n$ is bounded iff $\exists L$ st $|x| \leq L$ $\forall x \in X$.

Thm: (Heine-Borel) Any subspace $X \subset \mathbb{R}^n$ is cpt iff it is closed and bounded

↳ Proof next time.

Lemma: $f: X \to Y$ cts, $X =$ cpt $\Rightarrow$ $f(X) =$ cpt

Proof: Fix a cover $f(X) = \bigcup_\alpha f(X) \cap U_\alpha$, $U_\alpha =$ open in Y.

$\bigcup_\alpha f^{-1}(U_\alpha) =$ open cover for X

$\Rightarrow \exists\, U_1, \dots, U_n$ st $f^{-1}(U_1) \cup \dots \cup f^{-1}(U_n) = X$.

$\Rightarrow f(X) = f(\bigcup_i f^{-1}(U_i)) = \bigcup_i f(X) \cap U_i$

$\Rightarrow$ Admits finite subcover

Ex: $X =$ cpt $\Longrightarrow$ any cts fcn $f: X \to \mathbb{R}$ is bounded

Cor: $X \cong Y \Rightarrow X$ cpt iff Y cpt.

Defn: X = sequentially compact iff $\forall\, x_n$, $\exists$ subseq $x_{n_i} \longrightarrow x$.

Theorem: X = cpt + 1^{st} countable $\Rightarrow$ X = seq. compact ($\Leftarrow$ if X = metric space)

Proof: Spse the x_n do <u>not</u> have a convergent subsequence.

$\exists\, V_1 \supset \dots \supset V_n, \dots$ opens st $x \in \mathcal{U} \Rightarrow \exists\, i \gg 0$ st $V_i \subseteq \mathcal{U}$.

only finitely many x_n lie in some V_i for i suff. larger

↳ If not, then $\exists$ a subseq that converges to x.

$\Rightarrow \forall\, x \in X$, $\exists\, \mathcal{U}_x \ni x$ open st $\mathcal{U}_x \cap \bigcup_n x_n$ is finite.

$X = \bigcup_x \mathcal{U}_x \Rightarrow \exists\, \mathcal{U}_{x_1}, \dots, \mathcal{U}_{x_k}$ st $X = \bigcup_k \mathcal{U}_{x_k}$.

Since each $\mathcal{U}_{x_i}$ contains only finitely many x_n in the sequence

$\Rightarrow \exists$ finite # of x_n's $\Rightarrow\Leftarrow$. $\square$