Lecture #4 - September 19th, 2023

Wrapping up last time

Lemma: TFAE

① $f: X \to Y$ is cts

② $f^{-1}(C) =$ closed $\forall\ C \subseteq Y$ closed

③ for each $x \in X$ and $V \ni f(x)$ open, $\exists$ open $\mathcal{U}$ st $f(\mathcal{U}) \subseteq V$

④ $\forall\ A \subseteq X$, $f(\bar{A}) \subset \overline{f(A)}$

Cor: $f: \mathbb{R} \to \mathbb{R}$ cts iff $\forall\ x \in \mathbb{R}$, $\varepsilon > 0$, $\exists\ \delta > 0$ st

$$|x' - x| < \delta \implies |f(x') - f(x)| < \varepsilon$$

* See Lecture 3 notes for more general statement for metric spaces.

Proof: ($\Rightarrow$): Fix $x \in \mathbb{R}$, $\varepsilon > 0$.

f cts $\Rightarrow \exists\ x \in \mathcal{U} =$ open st $f(\mathcal{U}) \subseteq B_{f(x)}(\varepsilon)$

$\Rightarrow \exists\ \delta > 0$ st $B_\delta(x) \subset \mathcal{U}$. So $f(B_\delta(x)) \subset B_{f(x)}(\varepsilon)$

So $\big(|x' - x| < \delta \iff x' \in B_\delta(x)\big) \Rightarrow \big(|f(x') - f(x)| < \varepsilon \iff f(x') \in B_{f(x)}(\varepsilon)\big)$

($\Leftarrow$): Fix $V \subseteq \mathbb{R}$ open $\Rightarrow \exists\ \varepsilon > 0$ st $B_\varepsilon(f(x)) \subseteq V$

$\Rightarrow f(B_x(\delta)) \subseteq V$

$\square$

Lemma: (Pasteing Lemma): $X = A \cup B$ space, A, B closed.

If $f_A : A \to Y$ cts, $f_B : B \to Y$ cts, and $f_A = f_B$ on $A \cap B$

$$f: X \to Y, \quad f(x) = \begin{cases} f_A(x), & x \in A \\ f_B(x), & x \in B \end{cases}$$

is continuous

Proof: $C \subseteq Y$ closed, $f_A^{-1}(C) = f^{-1}(C) \cap A$ = closed in A

$\Rightarrow f_A^{-1}(C)$ = closed in X. Similarly, $f_B^{-1}(C)$ = closed in X

$\Rightarrow f^{-1}(C) = f_A^{-1}(C) \cup f_B^{-1}(C)$ = closed.

$\Rightarrow f$ cts. $\square$

Ex: ① $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} 0, & x \leq 0 \\ x, & x \geq 0 \end{cases}$

f is cts

② $\alpha : I \to X$ cts w/ $\alpha(1) = x$

$\beta : I \to X$ " " $\beta(0) = x$

$$\alpha * \beta := \gamma : I \to X, \quad \gamma(t) = \begin{cases} \alpha(2t), & 0 \leq t \leq 1/2 \\ \beta(2t-1), & 1/2 \leq t \leq 1 \end{cases}$$

Limits & Continuity

Question: $f: X \to Y$ cts iff $\forall\, x_n \to x$, $f(x_n) \to x$?

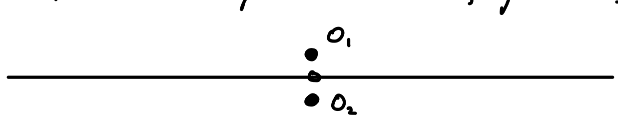
Defn: $\{x_n\}$ in X converges to $x \in X$ iff every open $x \in \mathcal{U}$, $\mathcal{U} \cap \{x_n\}_n$ contains all but finitely many x_n's. Write $x_n \to x$.

Warning: ① limits need not be unique.

↳ X = set w/ triv. top. and $x_n = x$ $\forall n$.

$\Rightarrow x_n \to y$ $\forall\, y \in X$.

② $\mathbb{R}_1 \sqcup \mathbb{R}_2 / \sim$, where $x \sim y$ iff $x \in \mathbb{R}_1$, $y \in \mathbb{R}_2$, $x = y$ w/ $\mathbb{R}_1$ identified w/ $\mathbb{R}_2$



$1/n \to 0_1$ and $1/n \to 0_2$

Lemma: $\{x_n\} \subseteq A$ and $x_n \to x \Rightarrow x \in \bar{A}$.

Proof: $\forall\, \mathcal{U} \ni x$ open, $\mathcal{U} \cap \{x_n\} \neq \emptyset \Rightarrow \mathcal{U} \cap A \neq \emptyset \Rightarrow x \in \bar{A}$ $\square$

Warning: $x \in \bar{A} \not\Rightarrow \exists\, x_n \in A$ w/ $x_n \longrightarrow x$.

↳ Exercise

Defn: A space is <u>first countable</u> iff $\forall\, x \in X$ $\exists$ countable # of opens $\mathcal{U}_1, \dots, \mathcal{U}_n, \dots$ st $\mathcal{U} \ni x$ open $\Rightarrow \mathcal{U}_i \subset \mathcal{U}$ for some i.

Ex: Every metric space is 1^{st} countable: $B_x(1/n)$

Lemma: $A \subseteq X$ = first countable, then $x \in \bar{A} \Rightarrow \exists\, x_n \in A$ w/ $x_n \longrightarrow x$.

Lemma: A space is <u>first countable</u> iff $\forall\, x \in X$ $\exists$ countable # of opens $V_1 \supset V_2 \supset \dots$ st $\mathcal{U} \ni x$ open $\Rightarrow V_i \subset \mathcal{U}$ for some i.

Proof: Let $x \in X$ and $\mathcal{U}_i$ as in defn.

$V_1 = \mathcal{U}_1$, $V_m = \mathcal{U}_{n_m}$ st $\mathcal{U}_{n_m} \subseteq \mathcal{U}_m \cap V_{m-1}$ $\square$

Proof: Consider $V_1, \dots, V_n, \dots$ as above.

$x \in \bar{A} \Rightarrow V_i \cap A \neq \emptyset \Rightarrow \exists\, x_i \in A \cap V_i$

$x \in \mathcal{U}$ = open, $\exists\, V_n \subset \mathcal{U} \Rightarrow x_k \in \mathcal{U}$ $\forall\, k \geq n$ $\square$

Lemma: f cts, then $\forall\, x_n \to x \Rightarrow f(x_n) \longrightarrow f(x)$

Proof: Spse $f(x_n) \not\to x \Rightarrow \mathcal{U}$ = open st $\mathcal{U} \cap \{f(x_n)\}$ is finite

$\Rightarrow f^{-1}(\mathcal{U}) \cap \{x_n\}$ is finite $\Rightarrow \Leftarrow$ $\square$

Lemma: X = first countable, $f: X \to Y$

If $\forall x_n \to X \Rightarrow f(x_n) \to f(x)$, then f cts

Proof: Consider $A \subseteq X$, $x \in \bar{A}$ and $x_n \in A$ st $x_n \to x$.

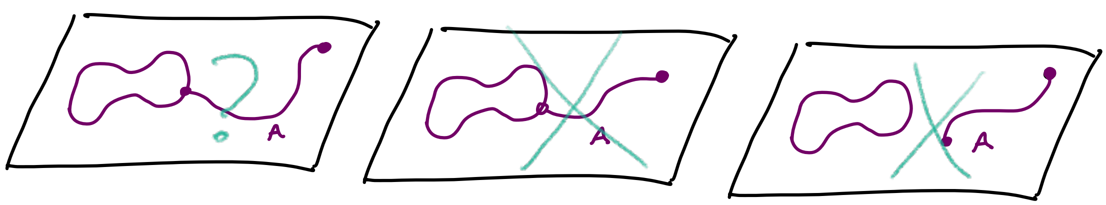
$f(x_n) \to f(x) \Rightarrow f(x) \in \overline{f(A)} \Rightarrow f(\bar{A}) \subseteq \overline{f(A)} \Rightarrow f$ cts $\square$

Connectedness

Defn: X = space is <u>connected</u> if the only subspaces of X that are both open + closed are ϕ, X.

Ex: ① X = set w/ discrete top.

$\Rightarrow \forall x \in X, \{x\} \subseteq X$ is open + closed $\Rightarrow X$ conn iff $|X| \leq 1$.

② 

Lemma: TFAE

(i) X = conn.

(ii) $X \neq U_0 \sqcup U_1$ st $U_i \subset X$ is open, $U_i \neq \phi$.

(iii) $X \neq C_0 \sqcup C_1$ " $C_i \subset X$ " closed, $C_i \neq \phi$.

Proof: (ii) $\Leftrightarrow$ (iii): $X = U_0 \sqcup U_1$, then U_i open $\Leftrightarrow$ U_i closed.

(i) $\Rightarrow$ (ii): Spse $X = U_0 \sqcup U_1$ st U_i = open and non-empty

$\Rightarrow X \supsetneq U_0$ = closed + open + non-empty

$\Rightarrow X$ <u>not</u> conn. $\Rightarrow \Leftarrow$

(ii) => (i): Spse X is not conn

$\Rightarrow \exists A \subsetneq X$ st A = open + closed + non-empty

$\Rightarrow B = X \setminus A$ = open + closed + non-empty

$\Rightarrow X = A \cup B$

$\Rightarrow$ not (ii) $\square$

Theorem: $[a,b] \subset \mathbb{R}$ is connected.

Fact: $C \subset \mathbb{R}$, spse $\exists L$ st $c \leq L \ \forall c \in C$.

Then $\exists$ least upper bound: $\exists L_0$ st if $c \leq L \ \forall c \in C \Rightarrow L_0 \leq L$ $\square$

Proof: Spse $[a,b] = A \cup B$ st A, B = closed + non-empty

WLOG, $b \in B \Rightarrow A$ has a lub $s < b$ $\quad s+\varepsilon < b$ for ε small.

Spse $s \in A$, A = open $\Rightarrow (s-\varepsilon, s+\varepsilon) \subseteq A \Rightarrow s \neq$ lub. $s \in B$

Note, s = limit pt of A. If not, then $\exists (s-\varepsilon, s+\varepsilon)$ st

$A \cap (s-\varepsilon, s+\varepsilon) = \phi \Rightarrow s \neq$ lub $\Rightarrow\Leftarrow$.

Since A closed $\Rightarrow s \in A$ $\Rightarrow\Leftarrow$

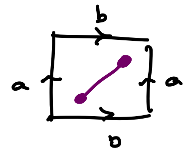
$\Rightarrow [a,b]$ = conn. $\square$

Defn: A path in X is a cts map $\gamma : [0,1] = I \longrightarrow X$.

γ is said to connect $\gamma(0)$ and $\gamma(1)$.

Defn: X = space is path-connected if $\forall a,b \in X$, $\exists$ path $\gamma : I \to X$ st $\gamma(0) = a$, $\gamma(1) = b$.

Ex:

1. $\mathbb{R}^n$ = path-connected $\rightsquigarrow$ $\gamma(t) = t \cdot x + (1-t) \cdot y$ conn x to y
2. S^n = path-connected $\rightsquigarrow$ $\gamma(t) = \dfrac{t \cdot x + (1-t) \cdot y}{|t \cdot x + (1-t) \cdot y|}$ conn x to y
3. Torus = path-connected $\rightsquigarrow$ 

Descends to cts map on the quotient.
4. $\mathbb{Q}$ $\neq$ path-conn.

Proposition: X = path-conn $\Rightarrow$ X = conn.

Proof: Spse $X = A \sqcup B$ st A, B open + closed

Fix $a \in A$, $b \in B$, $\gamma : I \to X$ w/ $\gamma(0) = a$, $\gamma(1) = b$.

$$I = \gamma^{-1}(X) = \gamma^{-1}(A \sqcup B) = \underbrace{\gamma^{-1}(A)}_{\substack{\text{open} \\ + \\ \text{closed}}} \sqcup \underbrace{\gamma^{-1}(B)}_{\substack{\text{open} \\ + \\ \text{closed}}} \Rightarrow\Leftarrow$$

$\square$