Lecture #3 - September 14th, 2023

Review of Quotient Spaces

Defn: An equiv rel. $\sim$ on X is a set of pairs $(x,y) \in X \times X$, written $x \sim y$ st

1. $x \sim x$
2. $x \sim y \Rightarrow y \sim x$
3. $x \sim y + y \sim z \Rightarrow x \sim z$

The equivalence class of x is $[x] = \{ y \in X \mid x \sim y \} \subseteq X$

$y \in [x]$ is said to be a representative of the equiv class

$X/\sim$ = set of equiv classes = $\bigcup_{[x]} [x]$ $\quad$ ($x \sim y \Leftrightarrow [x] = [y]$)

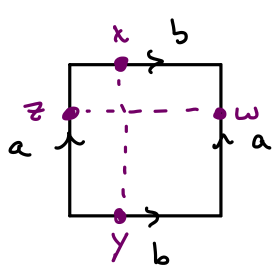
↳ identify pts that are equiv

$\exists$ can. surj map $q : X \to X/\sim$, $q(x) = [x]$

Defn: The quotient top on $X/\sim$: $U \subset X/\sim$ open $\Leftrightarrow$ $q^{-1}(U) =$ open

Ex:

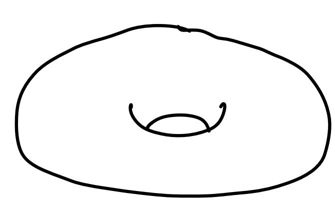
1. $X = D_1^2 \cup D_2^2$, $x \sim x$ and $x \sim y$ if $x \in \partial D_1$, $y \in \partial D_2$, $x = y$.
   $X/\sim = S^2 =$ sphere
2. $A \subseteq X$, $x \sim_A y \Leftrightarrow x = y$ or $x, y \in A$
   $X/A = X/\sim_A$
   $S^2 / S^1 = \text{equator} = S^2 \cup_{pt} S^2$
3. $[0,1]^2 \sim$ via pic



$x \sim y$

$z \sim w$

$[0,1]^2/\sim$ =



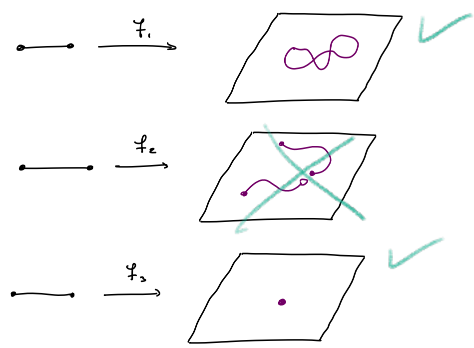
Continuous functions

Defn: X, Y = spaces, $f : X \to Y$ is <u>continuous</u> iff
$f^{-1}(U)$ = open $\forall\, U \subset Y$ open

Lemma: $f : X \to Y$ cts iff $f^{-1}(B)$ = open $\forall\, B$ in basis.

Proof: U = open $\Rightarrow$ $U = \bigcup_i B_i$ for B_i basic
$f^{-1}(U) = f^{-1}(\bigcup_i B_i) = \bigcup_i f^{-1}(B_i)$ = open. $\square$

Ex:

- 

Ex:

- X = discrete top on a set $\mathcal{O}_D$, Y = triv. top. on X $\mathcal{O}_T$
  $id : X \to Y$, continuous
  $id : Y \to X$, not continuous
- $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 7 + x$.
  $f^{-1}((a, b)) = (a-7, b-7)$ = open.

Lemma: $f : X \to Y$, $g : Y \to Z$ cts $\Rightarrow$ $g \circ f : X \to Z$ cts.

Proof: $(g \circ f)^{-1}(U) = f^{-1}(\underbrace{g^{-1}(U)}_{\text{open}})$ = open $\square$

Defn: $f: X \to Y$ cts is a <u>homeomorphism</u> iff $\exists$ cts inverse $g: Y \to X$.

We say X is <u>homeomorphic</u> to Y, $X \cong Y$.

Ex: $(-1,1)$ is homeomorphic to $\mathbb{R}$

$f: (-1,1) \to \mathbb{R}$, $f(x) = \frac{x}{1-|x|}$

$g: \mathbb{R} \to (-1,1)$, $g(y) = \frac{y}{1+|y|}$

One can check: $f \circ g = \mathbb{1} = g \circ f$

Use that f is surj + increasing

$f(x) - f(y) = \frac{x - x|y| - y|x| + y}{(1-|x|)(1-|y|)} \geq 0$

$$\left.\begin{aligned} f^{-1}((a,b)) &= \left\{ x \mid \tfrac{a}{1+|a|} < x < \tfrac{b}{1+|b|} \right\} = \text{open} \\ g^{-1}((a,b)) &= \left\{ y \mid \tfrac{a}{1-|a|} < y < \tfrac{b}{1-|b|} \right\} = \text{open} \end{aligned}\right\} \Rightarrow f, g = \text{cts}$$

Warning: $f: X \to Y$ cts + bijective $\not\Rightarrow$ f = homeo.

$\hookrightarrow$ $Y = \mathbb{R}$ w/ std. top.

$X = \mathbb{R}$ w/ discrete top.

$f: X \to Y$ identity map, $g: Y \to X$ inverse = identity.

$(a,b) \subset Y$, $f^{-1}((a,b)) = (a,b) = \bigcup_{x \in (a,b)} x$ = open

$g^{-1}(0) = 0 \neq$ open. $\square$

Ex: $S^n(r) = \{ x \in \mathbb{R}^{n+1} \mid |x|^2 = r \}$, $S^n(r^2) \cong S^n(1)$, $r^2 > 0$

$f: \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, $f(x) = r \cdot x$

$g: \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, $g(x) = \frac{x}{r}$

Want f, g = cts.

Would like to restrict f, g to $S^n(1), S^n(r^2)$ to get cts.

Lemma: $A \subseteq X$ subspace, $f: X \to Y$ cts $\Rightarrow$ $f|_A : A \to Y$ cts.

Proof: $f|_A^{-1}(\mathcal{U}) = A \cap f^{-1}(\mathcal{U})$ = open. $\square$

Cor: $A \subseteq X$ subspace. The inclusion $i: A \longrightarrow X$ is cts.

Lemma: $X, Y =$ spaces $\Rightarrow$ $\left.\begin{array}{l} pr_X : X \times Y \longrightarrow X, \ pr_X(x,y) = x \\ pr_Y : X \times Y \longrightarrow Y, \ pr_Y(x,y) = y \end{array}\right\}$ are cts.

Proof: $U \subseteq X$ open, $pr_X^{-1}(U) = U \times Y =$ open. $\square$

Lemma: $f: Z \longrightarrow X \times Y$, $f = (f_X, f_Y)$, $f_X : Z \to X$, $f_Y : Z \to Y$
f cts iff f_X, f_Y cts.

Proof: Check cts on basis: $U \times V \subseteq X \times Y$ basic open iff U, V open
$f^{-1}(U \times V) = f_X^{-1}(U) \cap f_Y^{-1}(V)$ $\square$

Cor: $f: \mathbb{R}^n \to \mathbb{R}^m$ cts iff $f = (f_1, \dots, f_m)$ & $f_i : \mathbb{R}^n \to \mathbb{R}$ cts

Defn: $q: X \to Y$ surj w/ $Y =$ quotient top. q is the quotient map.

Lemma: The quotient map $q: X \to Y$ is cont.

Proof: $U \subseteq Y$ open iff $q^{-1}(U) =$ open. $\square$

Lemma: Given quotient $q: X \to Y$, $f: Y \longrightarrow Z$ cts iff $f \circ q$ is cts.

Proof: f cts iff $U \subset Z$ open $\Rightarrow f^{-1}(U) =$ open
iff $U \subset Z$ open $\Rightarrow q^{-1}(f^{-1}(U)) =$ open
iff $f \circ q$ cts $\square$

Lemma: TFAE

1. $f: X \to Y$ is cts
2. $f^{-1}(C) = $ closed $\forall\, C \subseteq Y$ closed
3. for each $x \in X$ and $V \ni f(x)$ open, $\exists$ open $\mathcal{U}$ st $f(\mathcal{U}) \subseteq V$
4. $\forall\, A \subseteq X$, $f(\bar{A}) \subset \overline{f(A)}$

Proof: ① <=> ②:

$$f^{-}(C) = f^{-1}(Y \setminus \mathcal{U}) = f^{-1}(Y) \setminus f^{-1}(\mathcal{U}) = X \setminus \underbrace{f^{-1}(\mathcal{U})}_{\text{open}} = \text{closed.}$$

$$f^{-1}(\mathcal{U}) = f^{-1}(Y \setminus C) = f^{-1}(Y) - f^{-1}(C) = X \setminus \overbrace{f^{-1}(C)}^{\text{closed}} = \text{open}$$

① <=> ③:

(=>): Given $f(x) \in V = $ open, $\mathcal{U} := f^{-1}(V) = $ open

$$\Rightarrow f(\mathcal{U}) = f(f^{-1}(V)) \subseteq V$$

(<=): $V \subseteq Y$ open

if $f(X) \cap V = \emptyset \Rightarrow f^{-1}(V) = \emptyset = $ open

else $\exists\, x$ st $f(x) \in V \Rightarrow \exists\, \mathcal{U}_x \ni x$ open st $f(\mathcal{U}_x) \subseteq V$

$$\mathcal{U} = \bigcup_{f(x) \in V} \mathcal{U}_x \Rightarrow f^{-1}(V) = \mathcal{U} = \text{open}$$

② <=> ④:

(=>): $A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(\overline{f(A)}) = $ closed

$$\Rightarrow \bar{A} \subseteq f^{-1}(\overline{f(A)})$$

$$\Rightarrow f(\bar{A}) \subseteq \overline{f(A)}$$

(<=): $C \subseteq Y = $ closed. $f^{-1}(C) = A$

$$f(\bar{A}) \subseteq \overline{f(f^{-1}(C))} = C$$

$$\Rightarrow A = f^{-1}(C) = \bar{A} = \text{closed}$$

$\square$

Cor: $X, Y = $ metric spaces, $f: X \to Y$ is cts iff $\forall\, x \in X$, $\varepsilon > 0$ $\exists\, \delta > 0$ st $d(x', x) < \delta \Rightarrow d(f(x'), f(x)) < \varepsilon$.

Proof: $(\Rightarrow)$:

f cts $\Rightarrow \forall x \in X, \varepsilon > 0, \exists U \ni x$ st $f(U) \subseteq B_{f(x)}(\varepsilon)$.

U = open $\Rightarrow \exists \delta > 0$ st $B_x(\delta) \subseteq U$

$\Rightarrow f(B_x(\delta)) \subseteq B_{f(x)}(\varepsilon)$

So $|x' - x| < \delta \Rightarrow x' \in B_x(\delta) \Rightarrow f(x') \in B_{f(x)}(\varepsilon) \Rightarrow |f(x') - f(x)| < \varepsilon$

$(\Leftarrow)$:

Given $x \in X$, $V \ni f(x)$ open, $\exists B_{f(x)}(\varepsilon) \subseteq V$.

So $\exists B_x(\delta)$ st $f(B_x(\delta)) \subset B_{f(x)}(\varepsilon) \subseteq V$. $\square$

Fact:

1. $f(x_1, \ldots, x_n) = x_i$ is cts
2. $f: \mathbb{R}^n \to \mathbb{R}$, $g: \mathbb{R}^n \to \mathbb{R}$ cts $\Rightarrow f+g, f-g$ cts
3. " " " $\Rightarrow f \cdot g$ cts
4. " " " $\Rightarrow f/g$ cts over $\{x \in \mathbb{R}^n \mid g(x) \neq 0\}$