Lecture #24 - December 12th, 2023

Thm: If $X =$ "nice", then X has a universal cover.

Rem:

① $\tilde{X} = \{[\gamma] \mid \gamma : I \to X,\ \gamma(0) = x_0\}$ w/ $[\gamma] = [\delta]$ iff $\gamma \simeq \delta$ rel ∂I

$p : \tilde{X} \to X$, $p([\gamma]) = \gamma(1)$

② Given $[\gamma] \in \tilde{X}$, defn $\gamma_t = \begin{cases} \gamma(s), & 0 \leq s \leq t \\ \gamma(t), & \text{else} \end{cases}$

Note, $\tilde{\gamma} = t \mapsto [\gamma_t]$ is the lift of γ.

③ $\mathcal{U}_{[\gamma]} = \left\{ [\gamma \cdot \eta] \;\middle|\; \begin{array}{l} \mathcal{U} = \text{path-conn, open, } \pi_1(\mathcal{U}) \xrightarrow{0} \pi_1(X), \\ \eta : I \to \mathcal{U} \text{ w/ } \eta(0) = \gamma(1) \end{array} \right\}$

$p^{-1}(\mathcal{U}) = \bigcup_{[\gamma]} \mathcal{U}_{[\gamma]}$.

Prop: $X =$ "nice". $\forall$ subgroups $H \subseteq \pi_1(X, x_0)$, $\exists$ covering space $p_H : X_H \to X$ st $\text{Im}(p_{H*}) = H$.

Proof: Consider $\sim$ on $\tilde{X} = \{[\gamma] \mid \gamma : I \to X,\ \gamma(0) = x_0\}$ by

$[\gamma] \sim [\delta]$ iff $\gamma(1) = \delta(1)$ and $[\gamma \cdot \delta^{-1}] \in H$.

($\sim$ is an equiv. rel.) $\gamma \sim \gamma$ since $[\gamma \cdot \gamma^{-1}] = 0 \in H$

$\gamma \sim \delta \Rightarrow \delta \sim \gamma$ since $[\delta \cdot \gamma^{-1}] = [\gamma \cdot \delta^{-1}]^{-1} \in H$.

$\gamma \sim \delta, \delta \sim \varepsilon \Rightarrow \gamma \sim \varepsilon$ since $[\gamma \cdot \varepsilon^{-1}] = [\gamma \cdot \delta^{-1} \cdot \delta \cdot \varepsilon^{-1}] = [\gamma \cdot \delta^{-1}] \cdot [\delta \cdot \varepsilon^{-1}] \in H$

(Defn X_H) Let $X_H = \tilde{X} / \sim$.

Note, $[\gamma] \sim [\delta]$ iff $[\gamma \cdot \varepsilon] \sim [\delta \cdot \varepsilon]$ $\forall$ $\varepsilon : I \to X$ w/ $\varepsilon(0) = \gamma(1)$.

$\Rightarrow$ if $[\gamma] \sim [\delta]$, then $\sim$ identifies $\mathcal{U}_{[\gamma]}$ w/ $\mathcal{U}_{[\delta]}$.

$\Rightarrow$ $[\gamma] \mapsto \gamma(1)$ is a covering map over $\bigcup \mathcal{U}_{[\gamma]} / \sim \;\longrightarrow \mathcal{U}$.

(Compute $\operatorname{Im} p_*$) Fix $\tilde{x}_0 = [\text{constant path at } x_0] \in X_H$.

Spse $[\alpha] \in \pi_1(X, x_0)$.

As above, we have a lift $\tilde{\alpha} = \alpha_t$ w/ $\tilde{\alpha}(0) = \tilde{x}_0$, $\tilde{\alpha}(1) = [\alpha]$.

So $[\alpha] \in H$ iff $\tilde{x}_0 \sim [\alpha]$ iff $\tilde{\alpha}(0) = \tilde{\alpha}(1)$ iff $[\alpha] \in \operatorname{Im}(p_*)$ $\square$

Thm: Spse $X =$ "nice". $p_i : \tilde{X}_i \longrightarrow X$ for $i = 1,2$ are isom iff $\operatorname{Im}(p_{1*})$ is conjugate to $\operatorname{Im}(p_{2*})$.

Proof: ($\Rightarrow$) $\operatorname{Im}(p_{1*}) = \operatorname{Im}(p_{2*} \circ \phi_*) = p_{2*}(\pi_1(\tilde{X}_2, \phi(\tilde{x}_1)))$.

But $p_{2*}(\pi_1(\tilde{X}, \phi(\tilde{x}_1)))$ conj to $p_{2*}(\pi_1(\tilde{X}_2, \tilde{x}_2))$ since

$p_2 \circ \phi(\tilde{x}_1) = p_1(\tilde{x}_1) = x_0 = p_2(\tilde{X}_2)$

($\Leftarrow$) Spse $\operatorname{Im}(p_{1*}) = I_{[\alpha^{-1}]}(\operatorname{Im}(p_{2*}))$

Let $\tilde{\alpha} : I \longrightarrow \tilde{X}_2$ w/ $\tilde{\alpha}(0) = \tilde{x}_2$ be a lift of α.

Then $\pi_1(\tilde{X}_2, \tilde{\alpha}(1))$ is gen by $\tilde{\alpha}^{-1} \cdot \beta \cdot \tilde{\alpha}$ for $\beta \in \pi_1(\tilde{X}_2, \tilde{x}_2)$.

$p_{2*}(\tilde{\alpha}^{-1} \cdot \beta \cdot \tilde{\alpha}) = [\alpha]^{-1} \cdot [p \circ \beta] \cdot [\alpha]$.

So $p_{2*}(\pi_1(\tilde{X}_2, \tilde{\alpha}(1))) = p_{1*}(\pi_1(\tilde{X}_1, \tilde{x}_1))$

So $\exists$ lifts

$$
\begin{tikzcd}
 & \tilde{X}_2 \arrow[d, "p_2"] \arrow[dl, bend left, "\tilde{p}_2"] \\
\tilde{X}_1 \arrow[ur, bend left, "\tilde{p}_1"] \arrow[r, "p_1"'] & X
\end{tikzcd}
$$

w/ $\tilde{p}_1(\tilde{x}_1) = \tilde{\alpha}(1)$, $\tilde{p}_2(\tilde{\alpha}(1)) = \tilde{x}_1$.

Note, $\tilde{p}_2 \circ \tilde{p}_1$ is a lift of $p_1 : \tilde{X}_1 \longrightarrow X$ to $\tilde{X}_1$.

But $\mathbb{1} : \tilde{X}_1 \to \tilde{X}_1$ is also such a lift.

Since $\tilde{p}_2 \circ \tilde{p}_1(\tilde{x}_1) = \tilde{x}_1 \Rightarrow \tilde{p}_2 \circ \tilde{p}_1$, $\mathbb{1}$ agree at a point.

$\Rightarrow \tilde{p}_2 \circ \tilde{p}_1 = \mathbb{1}$.

Sim. $\tilde{p}_1 \circ \tilde{p}_2 = \mathbb{1}$.

So $\tilde{p}_1$ is the desired isomorphism. $\square$

Cor: $X =$ "nice"

$$\left\{ \begin{array}{l} \text{Conj. classes of} \\ \text{subgroups of } \pi_1(X, x_0) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Isom. classes of} \\ \text{path-conn covers } \tilde{X} \to X \end{array} \right\}$$

Regular Coverings

Defn: An self-isomorphism $\tilde{X} \to \tilde{X}$ of a covering space $p: \tilde{X} \to X$ is called a deck transformation.

$\hookrightarrow$ $\mathrm{Aut}(\tilde{X}) =$ group of deck transformations

Ex: $\mathrm{Aut}(\mathbb{R}) = \mathbb{Z}$

Lemma: $\phi \in \mathrm{Aut}(\tilde{X})$ is completely determined by $\phi(\tilde{x}_0)$ when $\tilde{X} =$ path-conn and locally-path-conn.

Proof: $\phi \in \mathrm{Aut}(X)$ is a lift of $p: \tilde{X} \to X$ to $\tilde{X}$. Any such lift is uniquely determined by where it sends a point. $\square$

Defn: A covering space is regular if for each $x \in X$ and $\tilde{x}, \tilde{x}' \in p^{-1}(x)$, $\exists\ \phi \in \mathrm{Aut}(\tilde{X})$ w/ $\phi(\tilde{x}) = \tilde{x}'$.

Thm: $X =$ "nice". $\tilde{X} \to X$ regular iff $\mathrm{Im}(p_*)$ is normal.
If normal, then $\mathrm{Aut}(\tilde{X}) = \pi_1(X, x_0) / \mathrm{Im}(p_*)$.

Proof: regular $\iff \forall\ \tilde{x}, \tilde{x}'\ \exists\ \phi$ w/ $\phi(\tilde{x}) = \tilde{x}'$

$\iff \exists$ lift

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ \tilde{X} \arrow[ur, "\phi"] \arrow[r, "p"'] & X \end{tikzcd}$$

$\iff p_*(\pi_1(\tilde{X}, \tilde{x})) = p_*(\pi_1(\tilde{X}, \tilde{x}'))\ \forall\ \tilde{x}, \tilde{x}'.$

$\iff p_*(\pi_1(\tilde{X}, \tilde{x}))$ is conj inv.

$\iff p_*(\pi_1(\tilde{X}, \tilde{x}))$ is normal.

$\pi_1(X, x_0) \longrightarrow \mathrm{Aut}(\tilde{X})$, $[\alpha] \mapsto \phi_\alpha$ st $\phi_\alpha(\tilde{\alpha}(0) = \tilde{x}_0) = \tilde{\alpha}(1)$.

Note, lift of $\alpha \cdot \beta$ is $\tilde{\alpha} \cdot \phi_\alpha(\tilde{\beta})$.

So $[\alpha] \cdot [\beta] \mapsto \phi_{\alpha \cdot \beta}(\tilde{x}_0) = \phi_\beta \circ \phi_\alpha(\tilde{x}_0) \implies \phi_{\alpha\beta} = \phi_\beta \cdot \phi_\alpha$.

$[\alpha] \in \mathrm{Ker}$ iff $[\alpha] \in p_*(\pi_1(\tilde{X}, \tilde{x}_0))$.

Fix a path $\gamma : I \to \tilde{X}$ st $\gamma(0) = \tilde{x}_0$, $\gamma(1) = \tilde{x}_1 \in p^{-1}(x_0)$.

Then $[p \circ \gamma] \in \pi_1(X, x_0)$ w/ lift γ. $\implies$ surj.

$\implies \pi_1(X, x_0) / \mathrm{Im}(p_*) = \mathrm{Aut}(\tilde{X})$. $\square$