Lecture #23 - December 7^{th}, 2023

Defn:

- X = locally path-conn if $\forall x \in \mathcal{U}$ open $\exists V$ = open + path-conn w/ $x \in V \subseteq \mathcal{U}$.
- X = semi-locally simply-connected if $\forall x \in X$, $\exists$ open $x \in \mathcal{U}$ st $\pi_1(\mathcal{U}, x) \longrightarrow \pi_1(X, x)$ is trivial.

Thm: If X = "nice", then X has a universal cover.

Lemma: $\mathcal{B} = \{ \mathcal{U} \subseteq X \mid \mathcal{U}$ = open, path-conn, $\pi_1(\mathcal{U}) \longrightarrow \pi_1(X)$ is trivial$\}$ is a basis for the top of X = "nice".

Proof: Since X = semi-locally sinply-conn, $\forall x \in X$, $\exists\ \mathcal{U} \in \mathcal{B}$ w/ $x \in \mathcal{U}$.
$\Rightarrow \mathcal{B}$ covers X.

Spse $x \in \mathcal{U} \cap V$ w/ $\mathcal{U}, V \in \mathcal{B}$.
Since X = locally path-conn, $\exists$ open $x \in W \subset \mathcal{U} \cap V$ w/ W = path-conn.
Note, $\pi_1(W, x) \to \pi_1(\mathcal{U}, x) \xrightarrow{\text{triv}} \pi_1(X, x) \Rightarrow \pi_1(W, x) \xrightarrow{\text{triv}} \pi_1(X, x) \Rightarrow W \in \mathcal{B}$.
If $x \in W \subseteq X$ open, NTS $\exists\ \mathcal{U} \in \mathcal{B}$ w/ $\mathcal{U} \subseteq W$.
X = semi-loc. simply-conn $\Rightarrow \exists\ x \in V$ open st $\pi_1(V, x) \to \pi_1(X, x)$ is zero.
X = loc. path-conn $\Rightarrow \exists\ x \in \mathcal{U} \subseteq V \cap W$ open + path-conn
$\Rightarrow \mathcal{U} \in \mathcal{B}$ w/ $\mathcal{U} \subseteq W$. $\square$

Proof: Step 1: Define $\tilde{X}$.

$\hookrightarrow \tilde{X} = \{ [\gamma] \mid \gamma : I \to X,\ \gamma(0) = x_0 \}$ w/ $[\gamma] = [\delta]$ iff $\gamma \simeq \delta$ rel ∂I

$p : \tilde{X} \to X$, $p([\gamma]) = \gamma(1)$ $\square$

Step 2: Define top. on $\tilde{X}$.

$\hookrightarrow$ Given $\mathcal{U} \in \mathcal{B}$, $[\gamma] \in \tilde{X}$ st $\gamma(1) \in \mathcal{U}$, define

$$\mathcal{U}_{[\gamma]} = \{ [\gamma \cdot \eta] \mid \eta : I \to \mathcal{U} \text{ w/ } \eta(0) = \gamma(1) \}$$

WTS, $\mathcal{U}_{[\gamma]}$ give a basis for a top. on $\tilde{X}$.

(*) $[\delta] \in \mathcal{U}_{[\gamma]} \Rightarrow \mathcal{U}_{[\gamma]} = \mathcal{U}_{[\delta]}$.

$\hookrightarrow$ $[\delta] = [\gamma \cdot \eta] \Rightarrow [\varepsilon] \in \mathcal{U}_{[\delta]}$ is $[\varepsilon] = [\gamma \cdot \eta \cdot \eta'] \in \mathcal{U}_{[\gamma]}$

Also, $[\gamma] = [\gamma \cdot \eta \cdot \eta^{-1}] = [\delta \cdot \eta^{-1}]$.

So $[\varepsilon] \in \mathcal{U}_{[\gamma]}$ is $[\varepsilon] = [\gamma \cdot \mu] = [\delta \cdot \eta^{-1} \cdot \mu] \in \mathcal{U}_{[\delta]}$

Note, $\tilde{X} = \bigcup_{[\gamma]} [\gamma] = \bigcup_{[\gamma]} \mathcal{U}_{[\gamma]} \Rightarrow$ cover.

$[\varepsilon] \in \mathcal{U}_{[\gamma]} \cap V_{[\delta]} \overset{(*)}{\Rightarrow} \mathcal{U}_{[\varepsilon]} = \mathcal{U}_{[\gamma]}$, $V_{[\varepsilon]} = V_{[\delta]}$.

$\exists\, W \in \mathcal{B}$ st $\varepsilon(1) \in W \subseteq \mathcal{U} \cap V$

Note, $[\varepsilon] \in W_{[\varepsilon]} \subseteq \mathcal{U}_{[\varepsilon]} \cap V_{[\varepsilon]} \subseteq \mathcal{U}_{[\gamma]} \cap V_{[\delta]}$ $\square$

Step 3: $p : \mathcal{U}_{[\gamma]} \longrightarrow \mathcal{U}$ is a homeo.

$\hookrightarrow$ (Surj.) $\mathcal{U}$ path-conn $\Rightarrow \forall x \in \mathcal{U}, \exists\, \eta : I \to \mathcal{U}$ w/ $\eta(0) = \gamma(1), \eta(1) = x$

So $p([\gamma \cdot \eta]) = x$.

(inj.) $p([\gamma \cdot \eta]) = p([\gamma \cdot \eta']) \Rightarrow \eta(1) = \eta'(1)$

But $\pi_1(\mathcal{U}) \to \pi_1(X)$ is triv., so $\eta \simeq \eta' \Rightarrow [\gamma \cdot \eta] = [\gamma \cdot \eta']$

(homeo) $\mathcal{U}$ has a basis $V \cap \mathcal{U}$ for $V \in \mathcal{B}$.

$\mathcal{U}_{[\gamma]}$ has a basis $V_{[\delta]} \cap \mathcal{U}_{[\gamma]}$ for $V \in \mathcal{B}$.

$p(V_{[\delta]} \cap \mathcal{U}_{[\gamma]}) = V \cap \mathcal{U}$

$p^{-1}(V \cap \mathcal{U}) \cap \mathcal{U}_{[\gamma]} = V_{[\delta]} \cap \mathcal{U}_{[\gamma]}$ for any $[\delta] \in \mathcal{U}_{[\gamma]}$ w/ $\delta(1) \in V$ since $V_{[\delta]} \subseteq \mathcal{U}_{[\delta]} \overset{(*)}{=} \mathcal{U}_{[\gamma]}$ and p maps $V_{[\delta]}$ bijectively to V.

Step 4: Show p is a covering space.

↳ • For $\mathcal{U} \in \mathcal{B}$, $p^{-1}(\mathcal{U}) = \bigcup_{[\gamma]} \mathcal{U}_{[\gamma]}$.

If $[\varepsilon] \in \mathcal{U}_{[\gamma]} \cap \mathcal{U}_{[\delta]} \overset{(*)}{\Longrightarrow} \mathcal{U}_{[\gamma]} = \mathcal{U}_{[\varepsilon]} = \mathcal{U}_{[\delta]}$

$\Rightarrow p^{-1}(\mathcal{U}) = \bigsqcup_{[\gamma]} \mathcal{U}_{[\gamma]}$

$\Rightarrow p =$ cts + covering map.

Step 5: $\tilde{X}$ is simply-connected.

↳ Let $[x_0]$ = class of constant path.

Given $[\gamma] \in \tilde{X}$, defn $\gamma_t = \begin{cases} \gamma(s) & , 0 \leq s \leq t \\ \gamma(t) & , \text{else} \end{cases}$

Note, $\tilde{\gamma} = t \mapsto [\gamma_t] \Rightarrow t \mapsto [\gamma_t]$ is cts path from $[x_0]$ to $[\gamma]$.

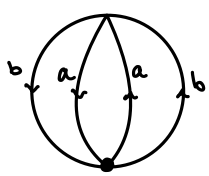
$\Rightarrow \tilde{X}$ = path-conn.

To show $\pi_1(\tilde{X}, [x_0]) = 0$, it STS $\text{Im}(p_*) = 0$.

$[\alpha] \in \text{Im}(p_*) \Rightarrow [x_0] = \tilde{\alpha}(0) = \tilde{\alpha}(1) = [\alpha] \Rightarrow [\alpha] = 0$ $\square$

Ex:

①



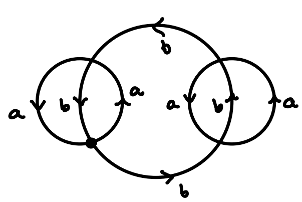
$\text{Aut} = \mathbb{Z}/2$

$\text{Im}(p_*) = \langle a^2, ab^{-1}, ab \rangle$

$\text{Im}(p_*)$ = normal

$\pi_1(S^1 \vee S^1)/\text{Im}(p_*) = \langle a, b \mid a^2, ab^{-1}, ab \rangle = \langle a \mid a^2 \rangle = \mathbb{Z}/2$

②



$\text{Aut} = \mathbb{Z}/2$

$\text{Im}(p_*) = \langle a^2, b^4, ab, ba^2b^{-1}, bab^{-2} \rangle$

$\text{Im}(p_*) \neq$ normal, $a^{-1}aba = ba \notin \text{Im}(p_*)$

$\pi_1(S^1 \vee S^1)/N(\text{Im}(p_*)) = \langle a, b \mid a^2, b^4, ab, ba^2b^{-1}, bab^{-2} \rangle = \mathbb{Z}/2$