Lecture #22 - December 5th, 2023

Defn: A covering space of X is a space $\tilde{X}$ and a map $p : \tilde{X} \to X$ st: $\forall x \in X$, $\exists$ open $x \in \mathcal{U}$ st $p^{-1}(\mathcal{U}) = \bigsqcup_\alpha \tilde{\mathcal{U}}_\alpha$ w/ $\tilde{\mathcal{U}}_\alpha$ open st $p|_{\tilde{\mathcal{U}}_\alpha} : \tilde{\mathcal{U}}_\alpha \to \mathcal{U}$ is a homeo.

- $\mathcal{U}$ is said to be evenly covered.
- $\tilde{\mathcal{U}}_\alpha$ are called sheets.
- When X = conn and p = surj, $|p_\alpha^{-1}(x)|$ = degree of the cover.

Thm: Given a hpty $f : Y \times I \to X$ and a lift $\tilde{f}_0 : Y \times \{0\} \to \tilde{X}$ of f_0, $\exists!$ hpty $\tilde{f} : Y \times I \to \tilde{X}$ that lifts f and $\tilde{f}|_{Y \times 0} = \tilde{f}_0$.

Prop:
1. $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ is injective.
2. $\mathrm{Im}(p_*) = \{ [\alpha] \in \pi_1(X, x_0) \mid \tilde{\alpha}(0) = \tilde{x}_0 = \tilde{\alpha}(1) \}$

Defn: Y = locally path-connected if $\forall y \in Y$ and $y \in \mathcal{U}$ open, $\exists$ open $y \in V \subseteq \mathcal{U}$ st V = path-conn.

Lemma: $f : Y \to X$ st Y = path-conn + locally path-connected
$\exists$ a lift $\tilde{f} : Y \to \tilde{X}$ w/ $f(y_0) = \tilde{x}_0$ iff $\mathrm{Im}(f_*) \subseteq \mathrm{Im}(p_*)$

Proof: If $\exists$ lift, $f_* = p_* \circ \tilde{f}_* \Rightarrow \mathrm{Im}(f_*) \subseteq \mathrm{Im}(p_*)$
Spse $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_0))$.
Pick a path $\gamma : I \to Y$ st $\gamma(0) = y_0$, $\gamma(1) = y$
Define $\tilde{f}(y) = \widetilde{f \circ \gamma}(1)$, where $\widetilde{f \circ \gamma}$ = lift of $f \circ \gamma$ w/ $\widetilde{f \circ \gamma}(0) = \tilde{x}_0$.

($\tilde{f}$ is well-defn): Spse δ is another path w/ $\delta(0) = y_0$, $\delta(1) = y$

$\Rightarrow [\gamma \cdot \delta^{-1}] \in \pi_1(Y, y_0)$.

$\Rightarrow \exists\ [\alpha] \in \pi_1(\tilde{X}, \tilde{x}_0)$ st $[p \circ \alpha] = [f \circ \gamma \cdot \delta^{-1}]$

$\Rightarrow p \circ \alpha \simeq f \circ (\gamma \cdot \delta^{-1})$ rel ∂I via a hpty H_t.

Let $\tilde{H}_t$ = lift w/ $\tilde{H}_0 = \alpha$.

$\tilde{H}_t(0) = \tilde{x}_0 = \tilde{H}_t(1)$ since $H_t(0) = x_0 = H_t(1)$

So by uniqueness, $\tilde{H}_1 = \widetilde{f \circ \gamma} \cdot \widetilde{(f \circ \delta)^{-1}}$.

$\Rightarrow \widetilde{f \circ \gamma}(1) = \tilde{H}_1(1/2) = \widetilde{f \circ \delta}(1)$.

($\tilde{f}$ is a lift): $p \circ \tilde{f}(y) = p \circ \widetilde{f \circ \gamma}(1) = f \circ \gamma(1) = f(y)$

($\tilde{f}$ is cts): Let $\tilde{f}(y) \in W \subseteq \tilde{X}$ open.

Let $f(y) \in \mathcal{U}$ = evenly covered by $\sqcup_\alpha \tilde{\mathcal{U}}_\alpha$. Spse $\tilde{f}(y) \in \tilde{\mathcal{U}}$.

STS $\exists\ V \ni y$ open st $\tilde{f}(V) \subseteq \tilde{\mathcal{U}} \cap W$.

Since f = cts, $\exists\ y \in V'$ open st $f(V) \subseteq p(\tilde{\mathcal{U}} \cap W)$

Let $y \in V \subseteq V'$ be a path-connected open.

Given $z \in V$, consider a path α from y to z in V.

So if γ is a path from y_0 to y $\Rightarrow$ $\gamma \cdot \alpha$ = path from y_0 to z.

Note, $\tilde{f}(z) = \widetilde{f \circ \gamma \cdot f \circ \alpha}(1) = \widetilde{f \circ \gamma} \cdot (p^{-1} \circ f \circ \alpha)(1) = p^{-1} \circ f(z)$.

$\Rightarrow \tilde{f}(V) \subseteq p|_{\tilde{\mathcal{U}}}^{-1}(f(V)) \subseteq p|_{\tilde{\mathcal{U}}}^{-1}(p(\tilde{\mathcal{U}} \cap W)) \subseteq \tilde{\mathcal{U}} \cap W$. $\square$

Universal covers

Defn:

- X = locally contractible if $\forall x \in X\ \exists$ open $x \in \mathcal{U}$ st $\mathcal{U} \simeq *$.
- X = locally simply-connected if $\forall x \in X\ \exists$ open $x \in \mathcal{U}$ st $\pi_1(\mathcal{U}) = 0$.
- X = semi-locally simply-connected if $\forall x \in X$, $\exists$ open $x \in \mathcal{U}$ st $\pi_1(\mathcal{U}, x) \longrightarrow \pi_1(X, x)$ is trivial.

Thm: CW-cpxes are locally contractible.

Ex: $X = \bigcup_n$ circle of radius $1/n$ centered at $(1/n, 0)$

$\hookrightarrow$ $X \neq$ semi-locally simply connected.

$\Rightarrow$ X = normal, but <u>not</u> a CW-cpx.

Prop: CW-cpxes are locally path-connected.

Defn: $p: \tilde{X} \to X$ is the universal cover of X if X = path-conn, $\tilde{X}$ = simply-conn.

Convention: X = path-connected, locally path-connected, semi-locally simply-connected.

$\Rightarrow$ X = "nice"

Thm:

① If X = "nice", then X has a universal cover.

② X = "nice". $\forall$ subgroups $H \subseteq \pi_1(X, x_0)$, $\exists$ covering space $p_H: X_H \to X$ st $\mathrm{Im}(p_{H*}) = H$.

Conjugacy Classes

Lemma: Given $g \in G$ = group, $\Phi_g : G \to G$, $\Phi_g(h) = ghg^{-1}$ is a self-isom.

Proof: $\Phi_g(h \cdot h') = ghh'g^{-1} = ghg^{-1}gh'g^{-1} = \Phi_g(h) \cdot \Phi_g(h')$ $\Rightarrow$ Φ_g = hom.

$\Phi_g \circ \Phi_{g^{-1}}(h) = h = \Phi_{g^{-1}} \circ \Phi_g(h)$ $\Rightarrow$ Φ_g = isom. $\square$

Defn: Two subgroups $H, H' \subseteq G$ are conjugate if $\exists g \in G$ w/ $\Phi_g(H) = H'$.

Classification Thm

Lemma: Spse $\tilde{X}$ = path-connected.
$p_*(\pi_1(\tilde{X},\tilde{x}_0))$ is conj. to $p_*(\pi_1(\tilde{X},\tilde{x}_1))$ for $\tilde{x}_0, \tilde{x}_1 \in p^{-1}(x_0)$.

Proof: Set $H_i = p_*(\pi_1(\tilde{X},\tilde{x}_i))$.
Fix a path $\gamma : I \to \tilde{X}$ w/ $\gamma(0) = \tilde{x}_0$, $\gamma(1) = \tilde{x}_1$. So $[p\circ\gamma] \in \pi_1(X, x_0)$.
Given $[\alpha] \in \pi_1(\tilde{X},\tilde{x}_1)$, $p_*[\gamma\cdot\alpha\cdot\gamma^{-1}] = [p\cdot\gamma]\cdot p_*([\alpha])\cdot[p\circ\gamma]^{-1} \in H_0$
$\Rightarrow \Phi_{[p\circ\gamma]}(H_1) \subseteq H_0$, $\Phi_{[p\circ\gamma^{-1}]}(H_0) \subseteq H_1$
But $\Phi_{[p\circ\gamma]}$ is the inverse of $\Phi_{[p\circ\gamma^{-1}]}$ $\Rightarrow \Phi_{[p\circ\gamma]}(H_1) = H_0$ $\square$

Defn: Two covering spaces $p_i : X_i \to X$ are isomorphic if $\exists$ homeo $\phi : X_1 \to X_2$ st $p_2\circ\phi = p_1$.

Exer: Isom. of covering spaces gives an equivalence relation.

Thm: (Classification of covering spaces thm): X = "nice", then $\exists$ a bij correspondence

$$\left\{\begin{array}{l}\text{Conj. classes of}\\ \text{subgrps of } \pi_1(X,x_0)\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{Isom. classes of}\\ \text{path-conn covers } \tilde{X}\to X\end{array}\right\}$$

Appendix

Thm: CW-cpxes are locally contractible.

Proof: Fix $x \in X = $ CW-cpx. Spse x is in an n-cell e^n_α.

Consider the char map $\Phi_\alpha : D^n_\alpha \longrightarrow X^n$.

So $\Phi_\alpha^{-1}(x) \in D^n_\alpha \setminus \partial D^n_\alpha$. Set $N^n_\varepsilon(x) = \Phi_\alpha(B_\varepsilon(x))$.

Note, $N^n_\varepsilon(x) = $ open + contractible.

Inductively, spse we have defn $N^m_\varepsilon(x) = $ open + contractible in X^m.

Consider char maps $\Phi_\alpha : D^{m+1}_\alpha \longrightarrow X^{m+1}$.

$N^{m+1}_\varepsilon(x) = \bigcup_\alpha \Phi_\alpha\big((1-\varepsilon, 1] \times \Phi_\alpha^{-1}(N^m_\varepsilon(x))\big)$, $\quad$ r-coord, θ-coord $\quad$ w/ polar coords for D^{m+1}_α.

$N^{m+1}_\varepsilon(X)$ is open in $X^m \sqcup_\alpha D^{m+1}_\alpha / \sim$ iff

$N^{m+1}_\varepsilon(x) \cap X^m$ open and $\Phi_\alpha^{-1}(N^{m+1}_\varepsilon(x))$ open $\forall \alpha$.

$\hookrightarrow$ $N^{m+1}_\varepsilon(x) \cap X^m = N^m_\varepsilon(x) = $ open

$\hookrightarrow$ $\Phi_\alpha^{-1}(N^{m+1}_\varepsilon(x)) = (1-\varepsilon, 1] \times \Phi_\alpha^{-1}(N^m_\varepsilon(x)) = $ open

$\Rightarrow$ $N^{m+1}_\varepsilon(x) = $ open

Note, $N^{m+1}_\varepsilon(x)$ defo retracts onto $N^m_\varepsilon(x)$ by pulling in the radial piece $(1-\varepsilon, 1]$ on $(1-\varepsilon, 1] \times \Phi_\alpha^{-1}(N^m_\varepsilon(x))$. $\Rightarrow$ $N^{m+1}_\varepsilon(x)$ contractible by induction. $\square$

Prop: CW-cpxes are locally path-connected.

Exer:

① A disjoint union of locally path-connected spaces is locally path-conn

② $X = $ loc. path-conn $\Rightarrow$ $X/\sim$ = locally path-connected.

Proof: The prop follows from inductively applying the exercise. $\square$