Lecture #21 - November 33th, 2023

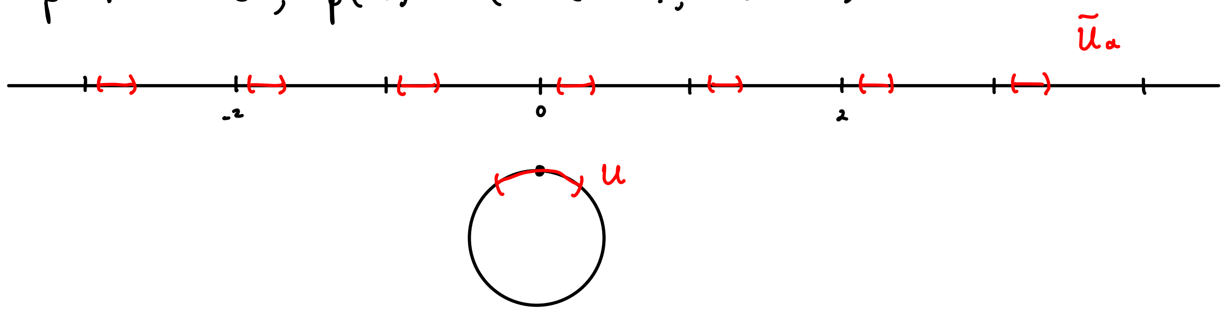
Covering Spaces

Defn: A covering space of X is a space $\tilde{X}$ and a map $p : \tilde{X} \to X$ st: $\forall x \in X$, $\exists$ open $x \in \mathcal{U}$ st $p^{-1}(\mathcal{U}) = \bigsqcup_\alpha \tilde{\mathcal{U}}_\alpha$ w/ $\tilde{\mathcal{U}}_\alpha$ open st $p|_{\tilde{\mathcal{U}}_\alpha} : \tilde{\mathcal{U}}_\alpha \to \mathcal{U}$ is a homeo.

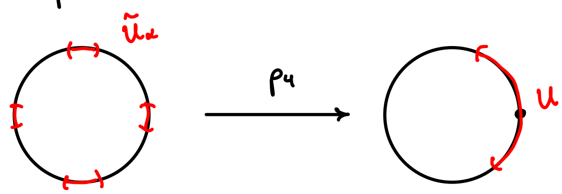
- $\mathcal{U}$ is said to be evenly covered.
- $\tilde{\mathcal{U}}_\alpha$ are called sheets.
- When X = conn and p = surj, $|p^{-1}(x)|$ = degree of the cover.

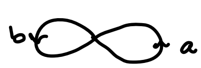
Example:

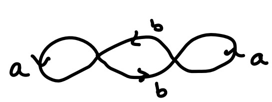
① $p : \mathbb{R} \to S^1$, $p(\lambda) = (\cos(2\pi\lambda), \sin(2\pi\lambda))$

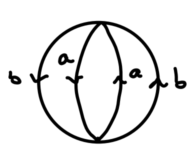


② $p_n : S^1 \to S^1$, $p(\cos(2\pi\theta), \sin(2\pi\theta)) = (\cos(2\pi n\theta), \sin(2\pi n\theta))$



Ex: Covers of $S^1 \vee S^1$ = 

①


②


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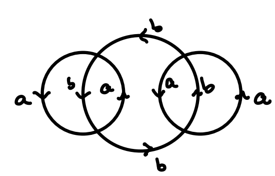

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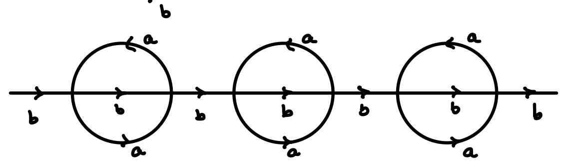

Homotopy Lifting Property

Notn: $p : \tilde{X} \to X$ is a covering space.

Defn: A <u>lift</u> of a map $f : Y \to X$ is a map $\tilde{f} : Y \to \tilde{X}$ st $f = p \circ \tilde{f}$.

Thm: Given a hpty $f : Y \times I \to X$ and a lift $\tilde{f}_0 : Y \times \{0\} \to \tilde{X}$ of f_0, $\exists!$ hpty $\tilde{f} : Y \times I \to \tilde{X}$ that lifts f w/ $\tilde{f}|_{Y \times 0} = \tilde{f}_0$.

Lemma: Spse $Y =$ conn, $f : Y \to X$. If two lifts $\tilde{f}_1, \tilde{f}_2 : Y \to \tilde{X}$ agree at <u>one</u> point of Y, then $\tilde{f}_1 = \tilde{f}_2$.

Proof of Lem: $\exists$ $f(y) \in \mathcal{U}$ open that is evenly covered. Write $p^{-1}(\mathcal{U}) = \bigsqcup_\alpha \tilde{\mathcal{U}}_\alpha$.

Spse $\tilde{f}_i(y) \in \tilde{\mathcal{U}}_i$.

Step 1: $\{ y \in Y \mid \tilde{f}_1(y) = \tilde{f}_2(y) \}$ is open

↳ $\tilde{f}_1(y) = \tilde{f}_2(y) \Rightarrow \tilde{\mathcal{U}}_1 = \tilde{\mathcal{U}}_2$

By cts, $\exists$ open $y \in V$ st $\tilde{f}_i(V) \subseteq \tilde{\mathcal{U}}_i$.

Note, $\tilde{f}_1|_V = p|_{\tilde{\mathcal{U}}_i}^{-1} \circ p|_{\tilde{\mathcal{U}}_i} \circ \tilde{f}_1|_V = p|_{\tilde{\mathcal{U}}_i}^{-1} \circ f|_V = \dots = \tilde{f}_2|_V$

Step 2: $\{ y \in Y \mid \tilde{f}_1(y) = \tilde{f}_2(y) \}$ is closed

↳ $\tilde{f}_1(y) \neq \tilde{f}_2(y) \Rightarrow \tilde{\mathcal{U}}_1 \neq \tilde{\mathcal{U}}_2$.

By cts, $\exists$ open $y \in V$ st $\tilde{f}_i(V) \subseteq \tilde{\mathcal{U}}_i$.

$\Rightarrow \tilde{f}_1|_V \neq \tilde{f}_2|_V$ $\forall$ points in V.

Since Y is conn $\Rightarrow$ $Y = \{ y \in Y \mid \tilde{f}_1(y) = \tilde{f}_2(y) \} \Rightarrow \tilde{f}_1 = \tilde{f}_2$ $\square$

Proof of Thm: Step 1: $\forall y \in Y$, $\exists\, y \in V$ open and $0 = t_0 \le t_1 \le \cdots \le t_n = 1$ st $f(V \times [t_i, t_{i+1}]) \subseteq \mathcal{U}_i$ w/ $\mathcal{U}_i$ evenly covered.

↳ Fix $y \in Y$. $\forall t$ $\exists$ open $f_t(y) \in \mathcal{U}_t$ that is evenly covered.

$\exists$ opens $V_t \subseteq Y$, $W_t \subseteq I$ st $(y,t) \in V_t \times W_t \subseteq f^{-1}(\mathcal{U}_t)$.

Note, $\bigcup_t V_t \times W_t$ cover $\{y\} \times I$ = cpt

$\Rightarrow \exists\, W_{s_0}, \dots, W_{s_m}$ $s_i \in I$ st $\bigcup_i V_{s_i} \times W_{s_i}$ covers $\{y\} \times I$.

Set $V = \bigcap_i V_{s_i}$.

Using the W_{s_i}'s, $\exists\, 0 = t_0 \le t_1 \le \cdots \le t_n = 1$ st

$f(V \times [t_i, t_{i+1}]) \subseteq \mathcal{U}_i$ w/ $\mathcal{U}_i$ evenly covered.

Step 2: $\exists$ hpty $\tilde{f}|_{V \times I} : V \times I \longrightarrow \tilde{X}$ that lifts $f|_{V \times I}$ w/ $\tilde{f}|_{V \times 0} = \tilde{f}_0|_V$.

↳ By induction, spse $\tilde{f}$ has been constructed over $V \times [0, t_i]$.

$f(V \times [t_i, t_{i+1}]) \subseteq \mathcal{U}$ = evenly covered.

Write $p^{-1}(\mathcal{U}) = \bigcup_\alpha \tilde{\mathcal{U}}_\alpha$ w/ homeos $p|_{\tilde{\mathcal{U}}_\alpha} : \tilde{\mathcal{U}}_\alpha \longrightarrow \mathcal{U}$.

By induction, $\tilde{f}|_{V \times \{t_i\}} : V \longrightarrow \tilde{\mathcal{U}}_\alpha$ for some fixed α.

Define $\tilde{f}|_{V \times [t_i, t_{i+1}]} = p|_{\tilde{\mathcal{U}}_\alpha}^{-1} \circ f|_{V \times [t_i, t_{i+1}]}$.

Note, $\tilde{f}|_{V \times [0, t_{i+1}]}$ = cts by the Pasting Lemma.

Note, $p \circ \tilde{f}|_{V \times [0, t_{i+1}]} = f|_{V \times [0, t_{i+1}]}$ since

(i) $p \circ \tilde{f}|_{V \times [0, t_i]} = f|_{V \times [0, t_i]}$

(ii) $p \circ \tilde{f}|_{V \times [t_i, t_{i+1}]} = p \circ p|_{\tilde{\mathcal{U}}_\alpha}^{-1} \circ f|_{V \times [t_i, t_{i+1}]} = f|_{V \times [t_i, t_{i+1}]}$.

Step 3: $\tilde{f}|_{V \times I}$ above is unique lift w/ $\tilde{f}|_{V \times 0} = \tilde{f}_0|_V$.

↳ $\tilde{f}|_{z \times I}$ is unique by the Lemma $\forall z \in V \Rightarrow \tilde{f}|_{V \times I}$ is unique.

Step 4: Construct $\tilde{f}$.

↳ $\forall y \in Y$, $\exists$ open V_y st $\tilde{f}|_{V_y \times I}$ exists and is unique.

By uniqueness, $\tilde{f}|_{V_y \times I}$ agree on overlaps of the V_y's.

So we obtain a cts map $\tilde{f} : Y \to \tilde{X}$.

As above, $\tilde{f}|_{z \times I}$ is unique $\forall z \in Y \Rightarrow \tilde{f}$ is unique. $\square$

Cor: Let $\gamma : I \to X$ be a path w/ $\gamma(0) = x_0$. Given $\tilde{x}_0 \in p^{-1}(x_0)$, $\exists!$ lift $\tilde{\gamma} : I \to \tilde{X}$ st $\tilde{\gamma}(0) = \tilde{x}_0$.

Prop: ① $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ is injective.

② $\mathrm{Im}(p_*) = \{ [\alpha] \in \pi_1(X, x_0) \mid \tilde{\alpha}(0) = \tilde{x}_0 = \tilde{\alpha}(1) \}$

Proof: ① Spse $p_*([\beta]) = 0$.

$\Rightarrow p \circ \beta \simeq$ constant rel ∂I.

Let $H_t : I \to X$ be the null-homotopy.

$\exists!$ lift $\tilde{H}_t : I \to \tilde{X}$ w/ $\tilde{H}_0 = \beta$.

Note, $x_0 = H_t(0) = p \circ \tilde{H}_t(0)$. So $\tilde{H}_t(0) = \tilde{x}_0$ is <u>a</u> lift.

By uniqueness, $\tilde{H}_t(0) = \tilde{x}_0$. Sim. $\tilde{H}_t(1) = \tilde{x}_0$.

Note, $x_0 = H_1(s) = p \circ \tilde{H}_1(s)$. So $\tilde{H}_1(s) = \tilde{x}_0$ is <u>a</u> lift.

By uniqueness, $\tilde{H}_1(s) = \tilde{x}_0$.

$\Rightarrow \beta \simeq$ constant rel ∂I

$\Rightarrow [\beta] = 0$

② $[\beta] \in \pi_1(\tilde{X}, \tilde{x}_0)$, then β is a lift of $p \circ \beta$.

$\Rightarrow \widetilde{p \circ \beta}(0) = \tilde{x}_0 = \widetilde{p \circ \beta}(1)$

$\Rightarrow \mathrm{Im}(p_*) \subseteq \{ [\alpha] \in \pi_1(X, x_0) \mid \tilde{\alpha}(0) = \tilde{x}_0 = \tilde{\alpha}(1) \}$

If $[\alpha] \in \pi_1(X, x_0)$ w/ $\tilde{\alpha}(0) = \tilde{x}_0 = \tilde{\alpha}(1)$, then $[\tilde{\alpha}] \in \pi_1(\tilde{X}, \tilde{x}_0)$

w/ $[\alpha] = [p \circ \tilde{\alpha}] = p_*([\tilde{\alpha}])$

$\Rightarrow \mathrm{Im}(p_*) \supseteq \{ [\alpha] \in \pi_1(X, x_0) \mid \tilde{\alpha}(0) = \tilde{x}_0 = \tilde{\alpha}(1) \}$ $\square$

Recall: $H \leq G =$ grp. H determines an equiv. rel $\sim_H$ by $g \sim_H g'$ iff $g \cdot g'^{-1} \in H$

Defn: The index of H is $[G : H] =$ # of equiv. classes determined by $\sim_H$.

Prop: If $X, \tilde{X}$ = path-conn, then $[\pi_1(X,x_0) : \mathrm{Im}(p_*)] = |p^{-1}(x_0)|$

Proof: We defn a bij. between $p^{-1}(x_0)$ and equiv. classes of $\sim_{\mathrm{Im}(p_*)}$

Fix $\tilde{x}_0 \in p^{-1}(x_0)$.

Defn $[\alpha]_{\sim \mathrm{Im}(p_*)} \longmapsto \tilde{\alpha}(1) \in p^{-1}(x_0)$ w/ $\tilde{\alpha}$ = lift of α w/ $\tilde{\alpha}(0) = \tilde{x}_0$.

(Well-defn): $\alpha \sim_{\mathrm{Im}(p_*)} \beta \Rightarrow \alpha \cdot \beta^{-1} \simeq p \circ \gamma$ rel ∂I for some $[\gamma] \in \pi_1(\tilde{X}, \tilde{x}_0)$

$\Rightarrow \gamma$ = lift of $\alpha \cdot \beta^{-1} \Rightarrow \gamma = \tilde{\alpha} \cdot \tilde{\beta}^{-1} \Rightarrow \tilde{\alpha}(1) = \gamma(1/2) = \tilde{\beta}^{-1}(0) = \tilde{\beta}(1)$

(Surj.) Let $\gamma : I \to \tilde{X}$ be a path w/ $\gamma(0) = \tilde{x}_0$, $\gamma(1) \in p^{-1}(x_0)$.

$\Rightarrow [p \circ \gamma] \in \pi_1(X, x_0)$ w/ $\widetilde{p \circ \gamma}(1) = \gamma(1)$.

(inj.) Spse $\tilde{\alpha}(1) = \tilde{\beta}(1)$ for $[\alpha], [\beta] \in \pi_1(X, x_0)$

$\Rightarrow [\tilde{\alpha} \cdot \tilde{\beta}^{-1}] \in \pi_1(\tilde{X}, \tilde{x}_0) \Rightarrow [\alpha \cdot \beta^{-1}] = [p \circ \tilde{\alpha} \cdot \tilde{\beta}^{-1}] \in \mathrm{Im}(p_*)$ $\square$