Lecture #20 - November 28th, 2023

The van Kampen Theorem

Thm: Spse $X = U \cup V$ w/

1. U, V = open + path-conn
2. $x_0 \in U \cap V$
3. $U \cap V$ = path-connected

We have maps

$$\begin{array}{ccc} \pi_1(U \cap V, x_0) & \xrightarrow{(j^U)_*} & \pi_1(U, x_0) \\ \downarrow{\scriptstyle (j^V)_*} & & \\ \pi_1(V, x_0) & & \end{array}$$

from the inclusions $j^U : U \cap V \to U$, $j^V : U \cap V \to V$.

Then $\pi_1(X, x_0) = \pi_1(U, x_0) *_{\pi_1(U \cap V, x_0)} \pi_1(V, x_0)$

Thm 1: Let Y = space obtained from X by attaching 2-cells:

X = Space, $\varphi_\alpha : \partial D^2_\alpha \to X$, $Y = X \sqcup_\alpha D^2_\alpha / \sim$

Let $\gamma_\alpha : I \to X$ st $\gamma_\alpha(0) = x_0$, $\gamma_\alpha(1) = \varphi_\alpha(0)$.

Let H = subgroup of $\pi_1(X, x_0)$ generated by $\{\gamma_\alpha \cdot \varphi_\alpha \cdot \gamma_\alpha^{-1}\}$.

$\Rightarrow$ $\pi_1(X) \to \pi_1(Y)$ is surj w/ $\ker = N(H)$

So $\pi_1(Y, x_0) = \pi_1(X, x_0) *_H 0 = \pi_1(X, x_0) / N(H)$

Thm 2: Let Y = space obtained from X by attaching n-cells w/ $n \geq 3$:

X = Space, $\varphi_\alpha : \partial D^n_\alpha \to X$, $Y = X \sqcup_\alpha D^n_\alpha / \sim$

$\Rightarrow$ $\pi_1(X) \to \pi_1(Y)$ is an isom.

Cor: X = CW-cpx, the inclusion $X^2 \hookrightarrow X$ induces an isom.

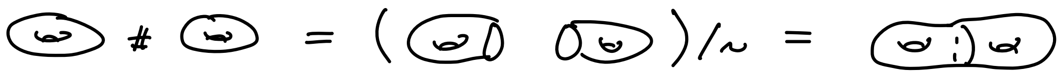
$$\pi_1(X^2, x_0) \xrightarrow{\cong} \pi_1(X, x_0)$$

Defn: Let Σ, Σ' = surfaces. The connect sum of Σ w/ Σ', denoted

$$\Sigma \# \Sigma' = (\Sigma \smallsetminus \mathrm{int}(D^2)) \sqcup (\Sigma' \smallsetminus \mathrm{int}(D^2)) / \sim$$

where $\sim$ identifies boundary points.

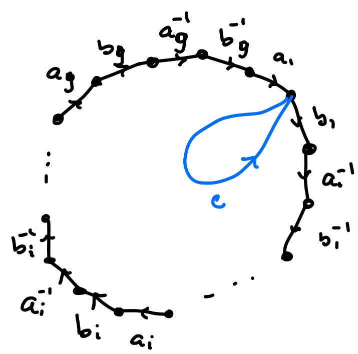
Picture:



Defn: The surface of genus g is the space $\underbrace{T^2 \# \cdots \# T^2}_{g\text{-times}} \# S^2$

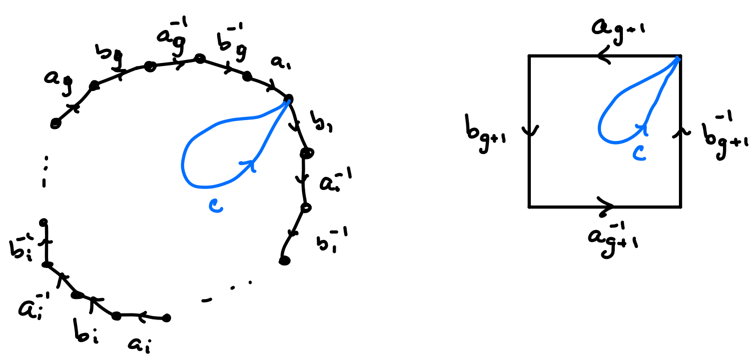
Prop: $\pi_1(\Sigma_g) = \langle a_1, b_1, \ldots, a_g, b_g \mid a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} \rangle$

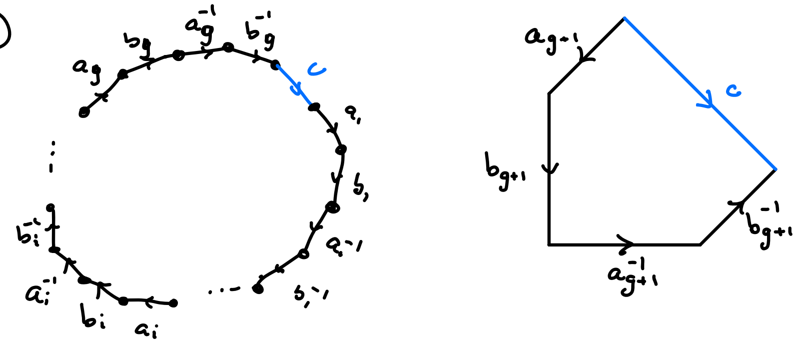
Proof: Σ_g has a CW-str. given by

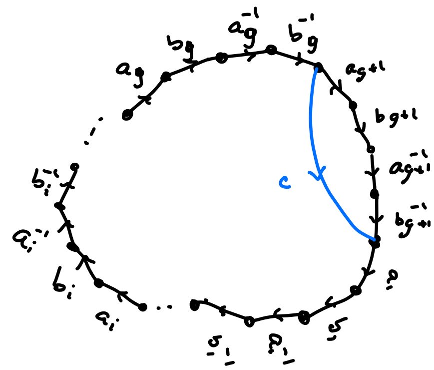


One constructs it inductively:

①



② 

③ 

So the CW-str. on Σ_g has $2g$ 1-cells all connected to same 0-cell.

So $\pi_1(\Sigma_g') = \langle a_1, b_1, \dots, a_g, b_g \rangle$.

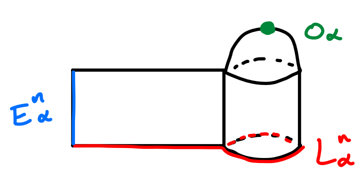
Now the attaching map is $\varphi = a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$.

Thm 1 $\Rightarrow$ $\pi_1(\Sigma_g) = \langle a_1, b_1, \dots, a_g, b_g \mid a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} \rangle$ $\square$

Set-up to prove Thms 1 + 2

Notn:

- Write $D_\alpha^n = \{ (r,\theta) \in I \times S^{n-1} \}$, ie, use polar coords.
- $S_\alpha = S_\alpha = I \times [0, 1/2]$
- $B_\alpha^n = D_\alpha^n \cup S_\alpha / ((t+1/2, 0) \sim (1,t) \text{ for } 0 \le t \le 1/2)$
- $L_\alpha^n = \partial D_\alpha^n \cup (0 \times [0,1/2]) / (1,0) \sim (1,0) = I \vee S^{n-1} \subseteq B_\alpha^n$

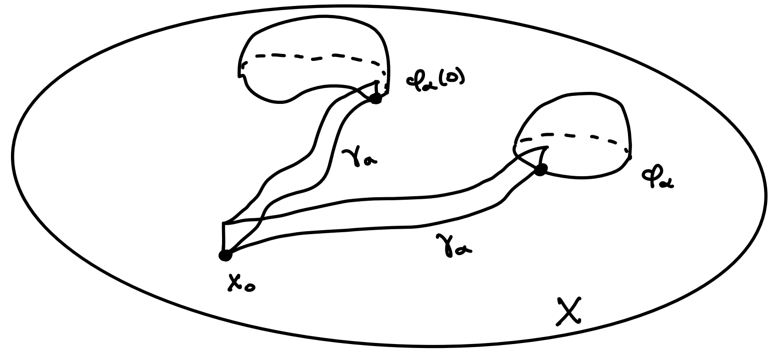
Picture: $B_\alpha^n =$ 

Notn: Consider attaching maps $\varphi_\alpha : \partial D^n_\alpha \longrightarrow X$.

Defn $\psi_\alpha : L^n_\alpha \longrightarrow X$ by, $\psi_\alpha = \gamma_\alpha \vee \varphi_\alpha$.

$\tilde{Y} = X \sqcup_\alpha B^n_\alpha \,/\, (\psi_\alpha(x) \sim x \,,\; E^n_\alpha \cong I \ni t \sim t \in E^n_\beta)$

Picture:

$\tilde{Y} =$ 

Notn:

- $\mathcal{U} = \tilde{Y} \setminus X$
- $V = \tilde{Y} \setminus \bigcup_\alpha O_\alpha$ w/ $O_\alpha =$ origin in D^n_α.

Lemma:

1. V defo retracts onto X
2. $\mathcal{U} \simeq *$
3. $\tilde{Y}$ defo retracts onto Y
4. $V \cap \mathcal{U} \simeq \bigvee_\alpha S^{n-1}$.

Proof: We will argue ① fairly explicitly, the details of the others are sim.

① Let $H_{\alpha,t}$ be the defo retract of $B^n_\alpha - O_\alpha$ onto L^n_α.

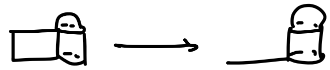
$$\begin{array}{ccccc} X \sqcup_\alpha (B^n_\alpha \setminus O_\alpha) & \xrightarrow{\mathbb{1} \sqcup_\alpha H_{\alpha,t}} & X \sqcup_\alpha (B^n_\alpha \setminus O_\alpha) & \xrightarrow{q} & X \sqcup_\alpha (B^n_\alpha \setminus O_\alpha)/\sim \\ \downarrow & & & & \| \\ V & & \xrightarrow{\quad H_t \quad} & & V \end{array}$$

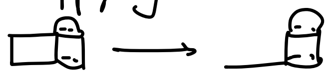
So $H_t([x]) = q \circ (\mathbb{1} \sqcup_\alpha H_{\alpha,t})(x)$ w/ $[x] =$ equiv class in quotient.

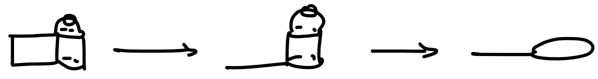
Note, H_t is well-defn + cts.

H_t is a defo retract of V onto X.

② $\mathcal{U} = \left(\bigsqcup_\alpha B^n_\alpha \setminus L^n_\alpha\right) / (E^n_\alpha \sim E^n_\beta) \overset{(A)}{\simeq} \bigvee_\alpha (I \vee D^n_\alpha) \simeq *$

w/ (A) given by applying the retracts 

③ Apply retracts 

④ Using retract ,

$\mathcal{U} \cap V \simeq \bigvee_\alpha I \vee \partial D^n_\alpha \simeq \bigvee_\alpha S^{n-1}$. $\square$

Pf thm 2: When $n \geq 3$, $\pi_1(\mathcal{U} \cap V) = 0$

$\Rightarrow \pi_1(\tilde{Y}) = \pi_1(\mathcal{U}) *_{\pi_1(\mathcal{U} \cap V)} \pi_1(V) = \pi_1(V)$

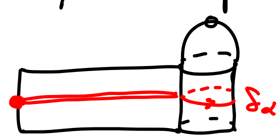
We have a commutative dgm:

$$\begin{tikzcd} \pi_1(X) \arrow[r] \arrow[d, "\simeq"'] \arrow[dr, phantom, "\circlearrowright"] & \pi_1(Y) \arrow[d, "\simeq"] \\ \pi_1(V) \arrow[r, "\simeq"] & \pi_1(\tilde{Y}) \end{tikzcd}$$

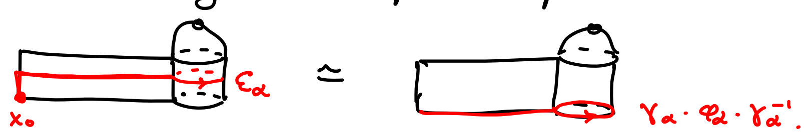
$\Rightarrow \pi_1(X) \to \pi_1(Y)$ is isom. $\square$

Pf thm 1: If $n = 2$, $\pi_1(\tilde{Y}) = \pi_1(\mathcal{U}) *_{\pi_1(\mathcal{U} \cap V)} \pi_1(V) = \pi_1(V) / N(i_*(\pi_1(\mathcal{U} \cap V)))$

$\pi_1(\mathcal{U} \cap V)$ is generated by the loops δ_α given by



$\Rightarrow i_*(\pi_1(\mathcal{U} \cap V))$ is generated by the loops ε_α



Consider the commutative dgm:

$$\begin{tikzcd} \pi_1(X) \arrow[r, "\Xi"] \arrow[d, "\simeq"'] \arrow[dr, phantom, "\circlearrowright"] & \pi_1(Y) \arrow[d, "\simeq"] \\ \pi_1(V) \arrow[r, "\Phi"] & \pi_1(\tilde{Y}) \end{tikzcd}$$

Note, Φ = surj $\Rightarrow$ Ξ = surj. and $\mathrm{Ker}(\Phi)$ is identified w/ $\mathrm{Ker}(\Xi)$

$\mathrm{Ker}(\Phi) = N(\gamma_\alpha \cdot \varphi_\alpha \cdot \gamma_\alpha^{-1}) \Rightarrow \mathrm{Ker}(\Xi) = N(\gamma_\alpha \cdot \varphi_\alpha \cdot \gamma_\alpha^{-1})$ $\square$