Lecture #2 - September 12th, 2023

Lemma: $\mathcal{U}_0, \mathcal{U}_1 \subset X$ = dense + open $\Rightarrow$ $\mathcal{U}_0 \cap \mathcal{U}_1$ = dense + open

Proof: NTS $X = \overline{\mathcal{U}_0 \cap \mathcal{U}_1}$

Spse $x \in X$

$\forall\, \mathcal{U} \ni x$ open, $\mathcal{U} \cap \mathcal{U}_0 \neq \emptyset$ since $\mathcal{U}_0$ dense

$\mathcal{U} \cap \mathcal{U}_0 \cap \mathcal{U}_1 \neq \emptyset$ since $\mathcal{U}_1$ dense

$\Rightarrow$ $x \in \overline{\mathcal{U}_0 \cap \mathcal{U}_1}$ $\square$

Metric spaces

Defn: A *metric* on a set X is a fcn $d: X \times X \to \mathbb{R}$ st

(i) $d(x,y) \geq 0$ w/ $d(x,y) = 0$ iff $x = y$

(ii) $d(x,y) = d(y,x)$

(iii) $d(x,y) + d(y,z) \geq d(x,z)$

Define $B_x(r) = \{ y \in X \mid d(x,y) < r \}$

Ex: $X = \mathbb{R}^n$, $d(x,y) = \left(\Sigma_i \, |x_i - y_i|^2 \right)^{1/2}$.

Defn: The *metric top.* on (X,d) is the top. generated by the basis

$$\mathcal{B} = \{ B_x(r) \mid x \in X,\ r \in \mathbb{R}_{>0} \}$$

Lemma: $\mathcal{B}$ is a basis

Proof: (i) $B_x(r)$ cover X

(ii) $x \in B_{y_0}(r_0) \cap B_{y_1}(r_1)$

$\ell = \min(r_0 - d(y_0,x),\ r_1 - d(y_1,x))$

Claim: $B_x(\varepsilon) \subseteq B_{y_0}(r_0) \cap B_{y_1}(r_1)$

Indeed, $d(y_0, z) \leq d(y_0, x) + d(x, z)$

$\leq d(y_0, x) + r_0 - d(y_0, x)$

$= r_0$ $\square$

Exercise: Diff. metric can give rise to same <u>or</u> different topologies

Subspaces

Defn: X = space w/ topology $\mathcal{O}$, $A \subseteq X$. The <u>subspace top.</u> on A is

$$\mathcal{O}_A = \{ A \cap U \mid U \in \mathcal{O} \}$$

We call A w/ this top. a <u>subspace</u> of X w/ top. $\mathcal{O}$.

Claim: $\mathcal{O}_A$ defines a topology

Proof: (i) $\phi = A \cap \phi \in \mathcal{O}_A$

$A = A \cap X \in \mathcal{O}_A$

(ii) $V_i \in \mathcal{O}_A$, $\bigcup_i V_i = \bigcup_i (A \cap U_i) = A \cap (\bigcup_i U_i) \in \mathcal{O}_A$

(iii) $\bigcap_i V_i = \bigcap_i (A \cap U_i) = A \cap (\bigcap_i U_i) \in \mathcal{O}_A$ $\square$

Rem: Any subset of $\mathbb{R}^n$ is a top. space.

Lemma: A basis $\mathcal{B}$ for $\mathcal{O}$ defines a basis $\mathcal{B}_A$ for $\mathcal{O}_A$ via

$$\mathcal{B}_A = \{ A \cap B \mid B \in \mathcal{B} \}$$

Proof: ① Show $\mathcal{B}_A$ is a basis

$$x \in (A \cap B') \cap (A \cap B'') \Rightarrow x \in B' \cap B''$$
$$\Rightarrow \exists\, x \in B \subset B' \cap B''$$
$$\Rightarrow x \in A \cap B \subseteq (A \cap B') \cap (A \cap B'')$$

② Spse $V \in \mathcal{O}_A$, $V = A \cap \mathcal{U}$, $x \in V$.

$\exists\, x \in B \subset \mathcal{U} \Rightarrow x \in A \cap B \subseteq V$

Spse $\forall x \in V \subset A$, $\exists\, x \in A \cap B_x \subseteq V$

$\Rightarrow V = \bigcup_x A \cap B_x = A \cap (\bigcup_x B_x) \in \mathcal{O}_A$ $\square$

Rem: If (X, d) = metric space, $A \subseteq X$, then (A, d_A) is metric space w/ $d_A(a_0, a_1) = d(a_0, a_1)$.

Lemma: (X, d) = metric space

Metric top on $A \subseteq X$ agrees w/ subspace top. of $A \subseteq X$.

Proof: Top. of X has basis $\mathcal{B} = \{B_x(r)\}_{x \in X}$

subspace top. of A has basis $\{A \cap B_x(r)\}_{x \in X} =: \mathcal{B}_S$

metric top. of A has basis $\mathcal{B}_M := \{B^A_x(r)\}_{x \in A} = \{A \cap B_x(r)\}_{x \in A}$

$\mathcal{B}_M \subseteq \mathcal{B}_S \Rightarrow \mathcal{U}$ open in metric $\Rightarrow$ open in subspace

$\mathcal{U} \ni x$ open in subspace $\Rightarrow B_y(r) \cap A \subseteq \mathcal{U}$ for some $y \in X$

$\Rightarrow \exists\, z \in B_y(r) \cap A \Rightarrow B^A_z(\varepsilon) \subset \mathcal{U}$ for $\varepsilon < \overset{\text{suff. small}}{\varepsilon} r \Rightarrow \mathcal{U}$ open in metric. $\square$

Defn: $A \subseteq X$ = space is <u>discrete</u> if its subspace top. is the discrete top.

Ex: Is $X = \{1/n\}_n \cup \{0\}$ discrete in $\mathbb{R}$?

<u>No</u>: $\{0\}$ is not open in X.

Spse $\{0\}$ is open $\Rightarrow \exists\, (-\varepsilon, \varepsilon)$ st $(-\varepsilon, \varepsilon) \cap X = \{0\}$.

But for each $\varepsilon > 0$, $\exists\, n$ st $1/n < \varepsilon$. $\Rightarrow \Leftarrow$ $\square$

<u>Warning:</u> $B \subseteq A$ open $\not\Rightarrow$ $B \subseteq X$ is open.

↳ $B = \mathbb{R} \subseteq \mathbb{R} = A$ open, but $A = \mathbb{R} \subseteq \mathbb{R}^2 = X$ is not open.

<u>Exercise:</u> $A \subseteq X$ open + $B \subseteq A$ open $\Rightarrow$ $B \subseteq X$ open

" closed " " closed " " closed.

<u>Warning:</u> $A \subseteq Y \subseteq X$, $\operatorname{int}(A)$ in $Y \neq Y \cap \operatorname{int}(A)$

↳ $(a,b) \times \{0\} = A \subseteq \mathbb{R} \times \{0\} = Y \subseteq \mathbb{R}^2 = X$

$\operatorname{int}(A)$ in $\mathbb{R}^2 = \varnothing$

$\operatorname{int}(A)$ in $Y = A$

<u>Exercise:</u> closure of A in Y = $Y \cap$ (closure of A in X).

Product Spaces

<u>Defn:</u> X, Y = spaces. The product top. on $X \times Y$ is the top w/ basis $\mathcal{U} \times V$ for $\mathcal{U} \subseteq X$, $V \subseteq Y$ opens.

<u>Lemma:</u> The above defn is well-defined.

<u>Proof:</u> (i) Cover since X, Y are opens

(ii) Spse $x \in \underbrace{\mathcal{U}_0 \times V_0 \cap \mathcal{U}_1 \times V_1}_{\text{Basic opens}} = (\mathcal{U}_0 \cap \mathcal{U}_1) \times (V_0 \cap V_1) \in$ Basis □

<u>Rem:</u>

- Inductively define $X_1 \times \dots \times X_n$.
- If $\mathcal{B}, \mathcal{C}$ are bases for $X, Y \Rightarrow \mathcal{B} \times \mathcal{C}$ is basis for $X \times Y$
  - ↳ Ex: $\mathbb{R}^2$ has basis $(a_1, b_1) \times (a_2, b_2)$.

Lemma: The metric top. on $\mathbb{R}^n$, $\mathcal{O}_\mu$, agrees w/ prod. top on $\mathbb{R}^n$, $\mathcal{O}_P$.

Proof: $\mathcal{U} \in \mathcal{O}_\mu$

$\iff \forall x \in \mathcal{U} \ \exists \ B_x(r) \subset \mathcal{U}$

$\iff \forall x \in \mathcal{U}, \ B_x(\varepsilon) \subset (x_1 - \varepsilon, x_1 + \varepsilon) \times \cdots \times (x_n - \varepsilon, x_n + \varepsilon) \subset B_x(r) \subset \mathcal{U}$

$\iff \mathcal{U} \in \mathcal{O}_P$ $\square$

Lemma: $A \subseteq X$, $B \subseteq Y$ are subspaces, then the subspace and prod. top. on $A \times B$ agree.

Proof: $\mathcal{O}_S$ is generated w/ basis $A \times B \cap \mathcal{U}_0 \times \mathcal{U}_1$ w/ $\mathcal{U}_0 \subset X$, $\mathcal{U}_1 \subset Y$ open

$\mathcal{O}_P$ is " " " $(A \cap \mathcal{U}_0) \times (B \cap \mathcal{U}_1)$ $\square$

Quotient top.

Defn: X = space, Y = set, $q : X \to Y$ surj.

The *quotient top* on Y is given by $\mathcal{U} \subseteq Y$ open iff $q^{-1}(\mathcal{U}) \subset X$ open.

Lemma: The above defn is well-defined.

Proof: (i) $q^{-1}(Y) = X$, $q^{-1}(\emptyset) = \emptyset \Rightarrow \emptyset, Y$ open

(ii) Spse $\mathcal{U}_i$ open

$q^{-1}(\bigcup_i \mathcal{U}_i) = \bigcup_i q^{-1}(\mathcal{U}_i)$ = open

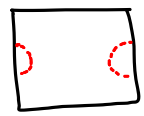
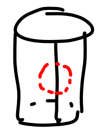
$q^{-1}(\bigcap_i \mathcal{U}_i) = \bigcap_i q^{-1}(\mathcal{U}_i)$ = open for $i \leq$ finite $\square$

Rem: $\sim$ = equiv rel. on X determines surj $q : X \to X/\sim$ = equiv. classes.

Ex: $A \subseteq X$ a subset. Define $x \sim_A y$ iff $x = y$ or $x, y \in A$.

$X/A = X/\sim_A$ = space w/ quotient topology.

Ex: $X/\partial \mathbb{D}^2 = S^2$

Ex: $X = [0,1]^2 =$  , $X/\sim\ =$ 

$(x_0, y_0) \sim (x_1, y_1)$ iff $x_0 = x_1$, $y_0 = y_1$ or $x_0 = 0$, $y_0 = y_1$, $x_1 = 1$.

Ex: $\mathbb{D}^2 = \{ x \in \mathbb{R}^2 \mid |x| \leq 1 \} \subseteq \mathbb{R}^2$.

$\partial \mathbb{D}^2 = \{ x \in \mathbb{R}^2 \mid |x| = 1 \} =: S^1$

$X = \mathbb{D}_1^2 \cup \mathbb{D}_2^2$, $x \in \partial \mathbb{D}_1^2$, $y \in \partial \mathbb{D}_2^2$ $x \sim y$ iff $x = y$ in $\mathbb{R}^2$.

$X/\sim\ =: S^2$