Lecture # 19 - November 21^{st}, 2023

The van Kampen Theorem

Thm: Spse $X = \mathcal{U} \cup V$ w/

① $\mathcal{U}, V$ = open + path-conn

② $x_0 \in \mathcal{U} \cap V$

③ $\mathcal{U} \cap V$ = path-connected

We have maps

$$\begin{tikzcd} \pi_1(\mathcal{U} \cap V, x_0) \arrow[r, "(j^{\mathcal{U}})_*"] \arrow[d, "(j^V)_*"'] & \pi_1(\mathcal{U}, x_0) \\ \pi_1(V, x_0) & \end{tikzcd}$$

from the inclusions $j^{\mathcal{U}}: \mathcal{U} \cap V \to \mathcal{U}$, $j^V: \mathcal{U} \cap V \to V$.

Then $\pi_1(X, x_0) = \pi_1(\mathcal{U}, x_0) *_{\pi_1(\mathcal{U} \cap V, x_0)} \pi_1(V, x_0)$

Cor: $\pi_1(S^n) = 0$ for $n \geq 2$.

Proof: Let $\mathcal{U}, V$ = open neighborhoods of northern/southern hemispheres.

So $\mathcal{U} \cap V \simeq S^{n-1}$.

By van Kampen, $\pi_1(S^n) = 0 *_{\pi_1(S^{n-1})} 0 = 0$ $\square$

Lemma: Spse $\exists$ $x_0 \in X$, $y_0 \in Y$ st $\exists$ opens $\mathcal{U} \ni x_0$, $V \ni y_0$ st x_0 is a defo retract of $\mathcal{U}$ and y_0 is a defo retract of V.

Then $\pi_1(X \vee Y) = \pi_1(X) * \pi_1(Y)$

Proof: ① $X \vee V \subseteq X \vee Y$ is homotopy equiv to X

$\mathcal{U} \vee Y \subseteq$ " " " " " Y

↳ Let $H_t : V \to V$ be the defo retract onto y_0

Define $\tilde{H}_t : X \vee V \to X \vee V$ by

$$\begin{tikzcd} X \sqcup V \arrow[r, "\mathbb{1} \sqcup H_t"] \arrow[d, "q"'] \arrow[dr, phantom, "\circlearrowleft"] & X \sqcup V \arrow[r, "q"] & X \vee V \\ X \vee V \arrow[urr, "\tilde{H}_t"'] & & \end{tikzcd}$$

Note, $\tilde{H}_t = $ cts since $q \circ \hat{H}_t = q \circ (\mathbb{1} \sqcup H_t) = $ cts

$\tilde{H}_0 = \mathbb{1}$, $\tilde{H}_1 = $ retract onto X, $\tilde{H}_t|_X = \mathbb{1}$

$\Rightarrow X \vee V$ defo retracts onto X.

Sim. $\mathcal{U} \vee Y$ " " " ".

② $\mathcal{U} \cup V \subseteq X \vee Y$ is contractible.

↳ Let $H_t : V \to V$, $G_t : \mathcal{U} \to \mathcal{U}$ be the defo retracts

Define $F_t : \mathcal{U} \vee V \to \mathcal{U} \vee \mathcal{U}$ by

$$\begin{tikzcd} \mathcal{U} \sqcup V \arrow[r, "G_t \sqcup H_t"] \arrow[d, "q"'] & \mathcal{U} \sqcup V \arrow[r, "q"] & \mathcal{U} \vee V \\ \mathcal{U} \vee V \arrow[urr, "F_t"'] & & \end{tikzcd}$$

As above, $F_t = $ cts

$F_0 = \mathbb{1}$, $F_1 = $ retract onto x_0, $F_t|_{x_0} = \mathbb{1}$

$\Rightarrow \mathcal{U} \vee V$ defo retracts onto x_0

③ van Kampen, $\pi_1(X \vee Y) = \pi_1(\mathcal{U} \vee Y) *_{\pi_1(\mathcal{U} \vee V)} \pi_1(X \vee V)$

$= \pi_1(Y) *_0 \pi_1(X)$

$= \pi_1(X) * \pi_1(Y)$ $\square$

Cor: $\pi_1(\bigvee_{i=1}^{n} S^1) = F_n$

Defn: A graph is a 1-dim'l CW cpx w/ finite # of vertices and edges.

↳ Γ = graph, $V(\Gamma)$ = # vertices / 0-cells

$E(\Gamma)$ = " edges / 1-cells

The Euler characteristic of Γ is $\chi(\Gamma) = V(\Gamma) - E(\Gamma)$

Defn: A spanning tree for Γ = graph is a subcomplex that is connected, contains every vertex, and is a tree.

Lemma: Every connected graph admits a spanning tree.

Proof: Induct: Spse every graph w/ n edges admits a spanning tree.

Let Γ have $n+1$ edges and let $e \subseteq \Gamma$ be an edge.

① $\Gamma \setminus e$ = connected

$\Rightarrow$ spanning tree for $\Gamma - e$ is a spanning tree for Γ

② $\Gamma \setminus e = \Gamma_0 \cup \Gamma_1$ w/ T_i = spanning tree for Γ_i

$\Rightarrow T_0 \cup T_1 \cup e$ = spanning tree for Γ $\square$

Lemma: Γ = connected graph, $\chi(\Gamma) \leq 1$ w/ equality iff Γ = tree.

Proof: Let T = spanning tree of Γ

$V(\Gamma) - E(\Gamma) = V(T) - E(T) - E(\Gamma \setminus T) = 1 - E(\Gamma \setminus T) \leq 1$ $\square$

Prop: Γ = connected graph, $\pi_1(\Gamma) = F_n$, where $n = 1 - \chi(\Gamma)$

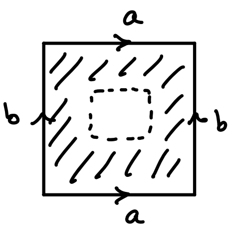
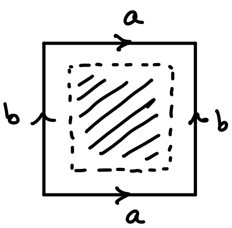
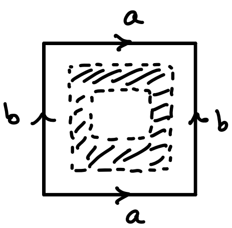
Proof: T = spanning tree $\simeq *$.

So $\Gamma \simeq \Gamma / T = \bigvee_{i=1}^{E(\Gamma - T)} S^1$

$\Rightarrow \pi_1(\Gamma) = F_n$ w/ $n = E(\Gamma - T) = 1 - \chi(\Gamma)$ $\square$

Lemma: $i : H \to G$, $G = \langle S \mid R \rangle$, $G *_H O = \langle S \mid R \cup i(H) \rangle$

Ex:

$\mathcal{U} =$  , $V =$  , $\mathcal{U} \cap V =$ 

$$\pi_1(T^2) = \pi_1(\mathcal{U}) *_{\pi_1(\mathcal{U} \cap V)} \pi_1(V) = \langle a, b \rangle *_{\pi_1(\mathcal{U} \cap V)} O = \langle a, b \mid aba^{-1}b^{-1} \rangle$$

Idea: 1-cells give generators, 2-cells give relations.

Thm 1: Let Y = space obtained from X by attaching 2-cells:

$$X = \text{Space}, \quad \varphi_\alpha : \partial D^2_\alpha \longrightarrow X, \quad Y = X \cup_\alpha D^2_\alpha / \sim$$

Let $\gamma_\alpha : I \to X$ st $\gamma_\alpha(0) = x_0$, $\gamma_\alpha(1) = \varphi_\alpha(0)$.

Let H = subgroup of $\pi_1(X, x_0)$ generated by $\{ \gamma_\alpha \cdot \varphi_\alpha \cdot \gamma_\alpha^{-1} \}$.

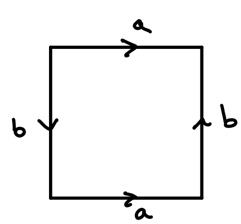
$\Rightarrow$ $\pi_1(X) \longrightarrow \pi_1(Y)$ is surj w/ $\text{Ker} = N(H)$

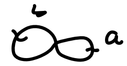
So $\pi_1(Y, x_0) = \pi_1(X, x_0) *_H O = \pi_1(X, x_0) / N(H)$

Rem:

- Thm 1 implies that if $\langle S \mid R \rangle$ is a presentation of $\pi_1(X)$, then $\langle S \mid R \cup \{ \gamma_\alpha \cdot \varphi_\alpha \cdot \gamma_\alpha^{-1} \} \rangle$ is a presentation of $\pi_1(Y)$.
- If $X = X^1$, $Y = X^2$ for a 2-dim'l CW-cpx, then $\pi_1(X)$ is a free group w/ some # of generators. The 2-cells give relations and determine the presentation for $\pi_1(X^2)$

Ex:

K = Klein bottle = 

$K^1 =$  $\Rightarrow$ $\pi_1(K^1) = \langle a, b \rangle$

$\Rightarrow$ $\pi_1(K) = \langle a, b \mid aba^{-1}b \rangle$

Ex: X = cpx obtained from attaching a single 2-cell to a circle $= \alpha$ by $\varphi = \alpha^n$.

$\pi_1(X) = \langle a \mid a^n \rangle = \mathbb{Z}/n$

Cor: For every group G, there is a 2-dim'l CW-cpx X w/ $\pi_1(X) = G$.

Proof: Write $G = \langle S \mid R \rangle$.

Set $X' = \bigvee_{s \in S} S^1_s$ and parameterize S^1_s by α_s.

For each $r = s_1^{\varepsilon_1} \cdots s_n^{\varepsilon_n} \in R$, attach a 2-cell by $\varphi_r = \alpha_{s_1}^{\varepsilon_1} \cdots \alpha_{s_n}^{\varepsilon_n}$.

By Thm 1, $\pi_1(X) = \langle S \mid R \rangle$ $\square$