Lecture #18 - November 16th, 2023

Fundamental Theorem of Algebra

Defn: A complex polynomial of deg n is a fcn $f: \mathbb{C} \to \mathbb{C}$ given by

$$f(z) = a_n z^n + \dots + a_1 z^1 + a_0$$

where $a_i \in \mathbb{C}$ and $a_n \neq 0$.

If $f(z_0) = 0$, then z_0 is a root of f.

Thm: Every degree $n > 0$ cpx poly. has n roots counting multiplicity.

Proof: Write $f = a_n \cdot z^n + \dots + a_0$.

$f(z_0) = 0$ iff $f(z_0)/a_n = 0 \Rightarrow$ WLOG $a_n = 1$

Spse that $f(z) \neq 0 \ \forall z$.

Consider $\gamma: I \longrightarrow \mathbb{C}^\times$, $\gamma(s) = f(e^{2\pi i s})/f(1)$

$\gamma(0) = \gamma(1) \Rightarrow [\gamma] \in \pi_1(\mathbb{C}^\times, 1)$

Defn $H_t : I \to \mathbb{C}^\times$, $H_t(s) = f(t \cdot e^{2\pi i \cdot s}) / f(t)$

Since $f \neq 0$, H_t is well-defn.

$H_t(0) = f(t)/f(t) = 1 = f(t)/f(t) = H_t(1)$

$H_0 = f(0)/f(0) = 1 =$ constant, $H_1 = \gamma$.

$\Rightarrow [\gamma] = [e] \in \pi_1(\mathbb{C}^\times, 1) = \pi_1(S^1) = \mathbb{Z}$

Define $g_t : \mathbb{C}^\times \longrightarrow \mathbb{C}^\times$,

$$g_t(z) = f(z/t) \cdot t^n = z^n + a_{n-1} \cdot t \cdot z^{n-1} + \dots + a_1 \cdot t^{n-1} \cdot z + a_0 \cdot t^n.$$

↳ Note, $t \neq 0$, $g_t(z) \neq 0$ since $f(z/t) \neq 0$

$t = 0$, $g_0(z) = z^n \neq 0$ since $z \in \mathbb{C}^\times$.

Defn $G_t : I \to \mathbb{C}^*$ by $G_t(s) = g_t(e^{2\pi i \cdot s})/g_t(1)$.

$G_t(0) = g_t(1)/g_t(1) = 1 = g_t(1)/g_t(1) = G_t(1)$

$G_0 = (e^{2\pi i \cdot s})^n = e^{2\pi i \cdot s \cdot n} = \cos(2\pi \cdot sn) + i \cdot \sin(2\pi \cdot sn)$

$G_1 = \gamma$

$\Rightarrow [\gamma] = n \in \mathbb{Z} = \pi_1(\mathbb{C}^*, 1) \Rightarrow \Leftarrow$.

$\Rightarrow f$ has a root, say z_0

Induct w/ $f(z)/(z - z_0) =$ poly of deg $n-1$. $\square$

Free Groups

Notn:

- S = set
- S^{-1} = set of symbols s^{-1} where $s \in S$.
- Write $(s^{-1})^{-1} = s$ for $s \in S$.

Defn:

- A <u>word</u> in S is a seq. $s_1 \dots s_n$ w/ $s_i \in S \cup S^{-1}$.
- A word $s_1 \dots s_n$ is <u>reduced</u> if $\forall i$
  1. $s_{i+1} \neq s_i^{-1}$
  2. $s_{i+1}^{-1} \neq s_i$

  ↳ We can write a word as $s_1^{\varepsilon_1} \dots s_n^{\varepsilon_n}$ st $s_i \in S$, $\varepsilon_i \in \{\pm 1\}$.
- $F(S)$ = set of reduced words
- $\cdot : F(S) \times F(S) \longrightarrow F(S)$,

$$(r_1^{\delta_1} \dots r_m^{\delta_m}) \cdot (s_1^{\varepsilon_1} \dots s_n^{\varepsilon_n}) = r_1^{\delta_1} \dots r_{m-k+1}^{\delta_{m-k+1}} s_k^{\varepsilon_k} \dots s_n^{\varepsilon_n}$$

where k is the smallest integer st $s_k^{\varepsilon_k} \neq r_{m-k+1}^{-\delta_{m-k+1}}$

↳ concatenate words and cancel symbols w/ their inverses

Prop: $(F(S), \cdot)$ = group. = free group on the set S.

Proof: ① (Unit) the unit is the empty word

② (inverses) $(s_1^{\varepsilon_1} \cdots s_n^{\varepsilon_n})^{-1} = s_n^{-\varepsilon_n} \cdots s_1^{-\varepsilon_1}$

③ (associative) $s \in S \cup S^{-1} \cup \{\text{empty word}\}$

Define $\sigma_s : F(S) \to F(S)$ w/ $s \in S \cup S^{-1}$ by

$$\sigma_s(s_1^{\varepsilon_1} \cdots s_n^{\varepsilon_n}) = \begin{cases} s\, s_1^{\varepsilon_1} \cdots s_n^{\varepsilon_n}, & s^{\varepsilon_n} \neq s^{-1} \\ s_2^{\varepsilon_2} \cdots s_n^{\varepsilon_n}, & s^{\varepsilon_n} = s^{-1} \end{cases}$$

Note, $(r_1^{\gamma_1} \cdots r_k^{\gamma_k}) \cdot (s_1^{\delta_1} \cdots s_n^{\delta_n} \cdot t_1^{\varepsilon_1} \cdots t_m^{\varepsilon_m})$

$= (\sigma_{r_1^{\gamma_1}} \circ \cdots \circ \sigma_{r_k^{\gamma_k}}) \circ (\sigma_{s_1^{\delta_1}} \circ \cdots \circ \sigma_{s_n^{\delta_n}})(t_1^{\varepsilon_1} \cdots t_m^{\varepsilon_m})$ fcn comp is assoc.

$= (\sigma_{r_1^{\gamma_1}} \circ \cdots \circ \sigma_{r_k^{\gamma_k}} \circ \sigma_{s_1^{\delta_1}} \circ \cdots \circ \sigma_{s_n^{\delta_n}})(t_1^{\varepsilon_1} \cdots t_m^{\varepsilon_m})$

$= (r_1^{\gamma_1} \cdots r_k^{\gamma_k} \cdot s_1^{\delta_1} \cdots s_n^{\delta_n}) \cdot (t_1^{\varepsilon_1} \cdots t_m^{\varepsilon_m})$ $\square$

Defn: A subset $S \subseteq G$ <u>generates</u> G if

$$\varphi : F(S) \to G, \quad \varphi(g_1^{\varepsilon_1} \cdots g_n^{\varepsilon_n}) = g_1^{\varepsilon_1} \cdot \ldots \cdot g_n^{\varepsilon_n}$$

is surj. We write $G = \langle S \rangle$

G is finitely gen. if one can take $|S| < \infty$.

Ex: ① $\mathbb{Z} = \langle 1 \rangle$

② $\mathbb{Z}/p = \langle n \rangle$ for any $0 < n < p$

Defn: $R \subseteq G$, the <u>normal</u> <u>closure</u> of R is the smallest normal subgrp that contains R, denote it by $N(R)$.

↪ $N(R)$ = take products/inverses of elms grg^{-1} w/ $g \in G$, $r \in R$

Defn: A <u>presentation</u> for G is a pair (S, R) st $R \subseteq F(S)$ w/

- $G = \langle S \rangle$
- $N(R) = \mathrm{Ker}(\varphi : F(S) \to G)$.

We write $G \cong \langle S \mid R \rangle \cong F(S)/N(R)$.

The elements in S are called generators.

" " " R " " relations

Note: $\langle S \mid R \rangle$ = group of words in S where any occurrance of a word in R is replaced by the trivial word.

Ex:

① $\mathbb{Z}/n = \langle a \mid a^n \rangle$

② $\mathbb{Z} \times \mathbb{Z} = \langle a, b \mid aba^{-1}b^{-1} \rangle$

③ $0 = \langle a \mid a \rangle$

④ $\mathbb{Z} = \langle a, b, c \mid abc^{-1}, ab^{-1} \rangle$

Ex: The Dihedral group: $D_n = \langle r, s \mid r^n, s^2, (sr)^2 \rangle$

Thm: Every group admits a presentation

Proof: Take $S = G$, $R = \mathrm{Ker}(\varphi : F(G) \to G)$.

Free Products

Setup: Let $i^0 : H \to G_0$, $i^1 : H \to G_1$ be homs

A group G is the amalgamated free product of G_0 and G_1 along H

① $\exists$ homs $\varphi_i : G_i \longrightarrow G$ w/ $\varphi_0 \circ i^0 = \varphi_1 \circ i^1$

② If $\psi_i : G_i \longrightarrow K$ are homs w/ $\psi_0 \circ i^0 = \psi_1 \circ i^1$, then $\exists!$ hom $\psi : G \longrightarrow K$ w/ $\psi \circ \varphi_i = \psi_i$

Pic:

$$
\begin{tikzcd}
H \arrow[r, "i^0"] \arrow[d, "i'"'] \arrow[dr, phantom, "\circlearrowright"] & G_0 \arrow[d, "\varphi_0"] \arrow[ddr, bend left, "\psi_0"] & \\
G_1 \arrow[r, "\varphi_1"] \arrow[drr, bend right, "\psi_1"'] & G \arrow[dr, dashed, "\exists!\ \psi"] & \\
 & & K
\end{tikzcd}
$$

Lemma: Such a G always exists: $G = \langle S_0 \cup S_1 \mid R_0 \cup R_1 \cup T \rangle$
where $G_i = \langle S_i \mid R_i \rangle$, $T = \{ i^0(h) \cdot i'(h^{-1}) \mid h \in H \}$

Proof: $\varphi_i : G_i \longrightarrow G$ by $\varphi_i(g) = g$ (obvious map)
Given $\psi_i : G_i \longrightarrow K$, defn $\psi : G \to K$ by

$$
\psi(s) = \begin{cases} \psi_0(s), & s \in S_0 \\ \psi_1(s), & s \in S_1 \end{cases}
$$

↳ Well-defined since $\psi_0 \circ i^0 = \psi_1 \circ i'$.

↳ ψ is forced by $\psi_i = \psi \circ \varphi_i$, so ψ is unique $\square$

Fact: A map $\tilde{\varphi} : G \to H$ st $N \subseteq \mathrm{Ker}(\tilde{\varphi})$ descends to a well-defined map $\varphi : G/N \to H$ by $\varphi([g]) = \tilde{\varphi}(g)$.

Lemma: Such a G is unique.

Proof: Let G' be another such group.

$$
\begin{tikzcd}
H \arrow[r, "i^0"] \arrow[d, "i^1"'] \arrow[dr, phantom, "\circlearrowright"] & G_0 \arrow[d, "\varphi_0"] \arrow[ddr, bend left, "\varphi_0'"] \arrow[dddrr, bend left=60, "\varphi_0"] & & \\
G_1 \arrow[r, "\varphi_1"] \arrow[drr, bend right, "\varphi_1'"'] \arrow[ddrrr, bend right=60, "\varphi_1"'] & G \arrow[dr, dashed, "\exists!\ \psi"] \arrow[ur, phantom, "\circlearrowright"] \arrow[d, phantom, "\circlearrowright"] & & \\
& & G' \arrow[dr, dashed, "\exists!\ \psi'"] & \\
& & & G
\end{tikzcd}
$$

Note

$$
\begin{tikzcd}
H \arrow[r, "i^0"] \arrow[d, "i^1"'] \arrow[dr, phantom, "\circlearrowright"] & G_0 \arrow[d, "\varphi_0"] \arrow[ddr, bend left, "\varphi_0"] & \\
G_1 \arrow[r, "\varphi_1"] \arrow[drr, bend right, "\varphi_1"'] & G \arrow[dr, dashed, "\exists!\ \mathbb{1}"] \arrow[ur, phantom, "\circlearrowright"] \arrow[d, phantom, "\circlearrowright"] & \\
& & G
\end{tikzcd}
$$

So by uniqueness, $\psi' \circ \psi = \mathbb{1} \Rightarrow G \cong G'$ by a unique isom $\square$

Defn: $G = G_0 *_H G_1$

Defn: $G, H =$ grps, The free product of two groups G, H is the group

$$G * H = G *_0 H = \langle S_G \cup S_H \mid R_G \cup R_H \rangle$$

Ex: $\underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{n\text{-times}} = F_n =$ free group on n letters.