

Fundamental Theorem of Algebra

Defn: A complex polynomial of deg n is a fun $f: \mathbb{C} \rightarrow \mathbb{C}$ given by

$$f(z) = a_n z^n + \dots + a_1 z^1 + a_0$$

where $a_i \in \mathbb{C}$ and $a_n \neq 0$.

If $f(z_0) = 0$, then z_0 is a root of f .

Thm: Every degree $n > 0$ cpx poly. has n roots counting multiplicity.

Proof: Write $f = a_n \cdot z^n + \dots + a_0$.

$f(z_0) = 0$ iff $f(z_0)/a_n = 0 \Rightarrow$ WLOG $a_n = 1$

Spse that $f(z) \neq 0 \forall z$.

Consider $\gamma: I \rightarrow \mathbb{C}^*$, $\gamma(s) = f(e^{2\pi i s})/f(1)$

$\gamma(0) = \gamma(1) \Rightarrow [\gamma] \in \pi_1(\mathbb{C}^*, 1)$

Defn $H_t: I \rightarrow \mathbb{C}^*$, $H_t(s) = f(t \cdot e^{2\pi i s})/f(t)$

Since $f \neq 0$, H_t is well-defn.

$H_t(0) = f(t)/f(t) = 1 = f(t)/f(t) = H_t(1)$

$H_0 = f(0)/f(0) = 1 = \text{constant}$, $H_1 = \gamma$.

$\Rightarrow [\gamma] = [e] \in \pi_1(\mathbb{C}^*, 1) = \pi_1(S^1) = \mathbb{Z}$

Define $g_t: \mathbb{C}^* \rightarrow \mathbb{C}^*$,

$$g_t(z) = f(z/t) \cdot t^n = z^n + a_{n-1} \cdot t \cdot z^{n-1} + \dots + a_1 \cdot t^{n-1} \cdot z + a_0 \cdot t^n.$$

\hookrightarrow Note, $t \neq 0$, $g_t(z) \neq 0$ since $f(z/t) \neq 0$

$t = 0$, $g_0(z) = z^n \neq 0$ since $z \in \mathbb{C}^*$.

Defn $G_t: \mathbb{I} \rightarrow \mathbb{C}^*$ by $G_t(s) = g_t(e^{2\pi i \cdot s}) / g_t(1)$.

$$G_t(0) = g_t(1) / g_t(1) = 1 = g_t(1) / g_t(1) = G_t(1)$$

$$G_0 = (e^{2\pi i \cdot s})^n = e^{2\pi i \cdot s \cdot n} = \cos(2\pi \cdot sn) + i \cdot \sin(2\pi \cdot sn)$$

$$G_1 = \gamma$$

$$\Rightarrow [\gamma] = n \in \mathbb{Z} = \pi_1(\mathbb{C}^*, 1) \Rightarrow \Leftarrow$$

$\Rightarrow f$ has a root, say z_0

Induct w/ $f(z) / (z - z_0) = \text{poly of deg } n-1$. □

Free Groups

Notn:

- $S = \text{set}$
- $S^{-1} = \text{set of symbols } s^{-1} \text{ where } s \in S$.
- Write $(s^{-1})^{-1} = s$ for $s \in S$.

Defn:

- A word in S is a seq. $s_1 \dots s_n$ w/ $s_i \in S \cup S^{-1}$.
- A word $s_1 \dots s_n$ is reduced if $\forall i$

$$\textcircled{1} \quad s_{i+1} \neq s_i^{-1}$$

$$\textcircled{2} \quad s_{i+1}^{-1} \neq s_i$$

\hookrightarrow We can write a word as $s_1^{\epsilon_1} \dots s_n^{\epsilon_n}$ st $s_i \in S, \epsilon_i \in \{\pm 1\}$.

- $F(S) = \text{set of reduced words}$
- $\cdot : F(S) \times F(S) \rightarrow F(S)$,

$$(r_1^{\delta_1} \dots r_m^{\delta_m}) \cdot (s_1^{\epsilon_1} \dots s_n^{\epsilon_n}) = r_1^{\delta_1} \dots r_{m-k+1}^{\delta_{m-k+1}} s_k^{\epsilon_k} \dots s_n^{\epsilon_n}$$

where k is the smallest integer st $s_k^{\epsilon_k} \neq r_{m-k+1}^{-\delta_{m-k+1}}$

\hookrightarrow concatenate words and cancel symbols w/ their inverses

Prop: $(F(S), \cdot) = \text{group} = \text{free group on the set } S.$

Proof: ① (Unit) the unit is the empty word

② (inverses) $(s_1^{\epsilon_1} \dots s_n^{\epsilon_n})^{-1} = s_n^{-\epsilon_n} \dots s_1^{-\epsilon_1}$

③ (associative) $s \in S \cup S^{-1} \cup \{\text{empty word}\}$

Define $\sigma_s : F(S) \rightarrow F(S)$ w/ $s \in S \cup S^{-1}$ by

$$\sigma_s(s_1^{\epsilon_1} \dots s_n^{\epsilon_n}) = \begin{cases} s s_1^{\epsilon_1} \dots s_n^{\epsilon_n} & , s^{\epsilon_n} \neq s^{-1} \\ s_2^{\epsilon_2} \dots s_n^{\epsilon_n} & , s^{\epsilon_n} = s^{-1} \end{cases}$$

Note, $(r_1^{\gamma_1} \dots r_k^{\gamma_k}) \cdot (s_1^{\delta_1} \dots s_n^{\delta_n} \cdot t_1^{\zeta_1} \dots t_m^{\zeta_m})$

$$= (\sigma_{r_1^{\gamma_1}} \circ \dots \circ \sigma_{r_k^{\gamma_k}}) \circ (\sigma_{s_1^{\delta_1}} \circ \dots \circ \sigma_{s_n^{\delta_n}}) (t_1^{\zeta_1} \dots t_m^{\zeta_m}) \quad \left. \begin{array}{l} \text{fun comp} \\ \text{is assoc.} \end{array} \right\}$$

$$= (\sigma_{r_1^{\gamma_1}} \circ \dots \circ \sigma_{r_k^{\gamma_k}} \circ \sigma_{s_1^{\delta_1}} \circ \dots \circ \sigma_{s_n^{\delta_n}}) (t_1^{\zeta_1} \dots t_m^{\zeta_m})$$

$$= (r_1^{\gamma_1} \dots r_k^{\gamma_k} \cdot s_1^{\delta_1} \dots s_n^{\delta_n}) \cdot (t_1^{\zeta_1} \dots t_m^{\zeta_m}) \quad \square$$

Defn: A subset $S \subseteq G$ generates G if

$$\varphi : F(S) \rightarrow G, \quad \varphi(g_1^{\epsilon_1} \dots g_n^{\epsilon_n}) = g_1^{\epsilon_1} \cdot \dots \cdot g_n^{\epsilon_n}$$

is surj. We write $G = \langle S \rangle$

G is finitely gen. if one can take $|S| < \infty$.

Ex: ① $\mathbb{Z} = \langle 1 \rangle$

② $\mathbb{Z}/p = \langle n \rangle$ for any $0 < n < p$

Defn: $R \subseteq G$, the normal closure of R is the smallest normal subgroup that contains R , denote it by $N(R)$.

$\hookrightarrow N(R) = \text{take products/inverses of elms } g r g^{-1} \text{ w/ } g \in G, r \in R$

Defn: A presentation for G is a pair (S, R) st $R \subseteq F(S)$ w/

• $G = \langle S \rangle$

• $N(R) = \text{Ker}(\varphi: F(S) \rightarrow G)$.

We write $G \cong \langle S | R \rangle \cong F(S)/N(R)$.

The elements in S are called generators.

" " " R " " " relations

Note: $\langle S | R \rangle =$ group of words in S where any occurrence of a word in R is replaced by the trivial word.

Ex: ① $\mathbb{Z}/n = \langle a | a^n \rangle$

② $\mathbb{Z} \times \mathbb{Z} = \langle a, b | aba^{-1}b^{-1} \rangle$

③ $0 = \langle a | a \rangle$

④ $\mathbb{Z} = \langle a, b, c | abc^{-1}, ab^{-1} \rangle$

Ex: The Dihedral group: $D_n = \langle r, s | r^n, s^2, (sr)^2 \rangle$

Thm: Every group admits a presentation

Proof: Take $S = G$, $R = \text{Ker}(\varphi: F(G) \rightarrow G)$.

Free Products

Setup: Let $i_0: H \rightarrow G_0$, $i_1: H \rightarrow G_1$ be homs

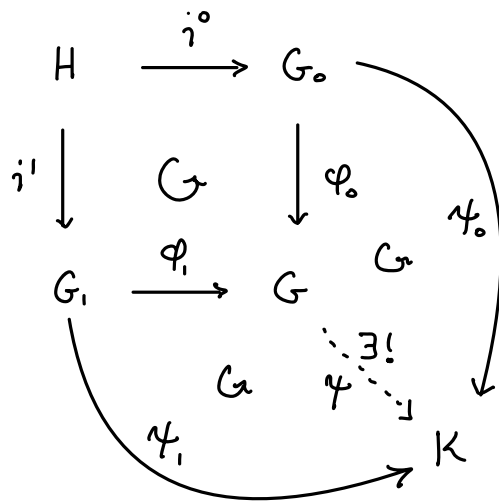
A group G is the amalgamated free product of G_0 and G_1 along H

① \exists homs $\varphi_i: G_i \rightarrow G$ w/ $\varphi_0 \circ i_0 = \varphi_1 \circ i_1$

② If $\psi_i: G_i \rightarrow K$ are homs w/ $\psi_0 \circ i_0 = \psi_1 \circ i_1$, then $\exists!$

hom $\psi: G \rightarrow K$ w/ $\psi \circ \varphi_i = \psi_i$

Pic:



Lemma:

Such a G always exists: $G = \langle S_0 \cup S_1 \mid R_0 \cup R_1 \cup T \rangle$
where $G_i = \langle S_i \mid R_i \rangle$, $T = \{i^0(h) \cdot i^1(h^{-1}) \mid h \in H\}$

Proof:

$\phi_i: G_i \rightarrow G$ by $\phi_i(g) = g$ (obvious map)

Given $\psi_i: G_i \rightarrow K$, defn $\psi: G \rightarrow K$ by

$$\psi(s) = \begin{cases} \psi_0(s), & s \in S_0 \\ \psi_1(s), & s \in S_1 \end{cases}$$

\hookrightarrow Well-defined since $\psi_0 \circ i^0 = \psi_1 \circ i^1$.

\hookrightarrow ψ is forced by $\psi_i = \psi \circ \phi_i$, so ψ is unique \square

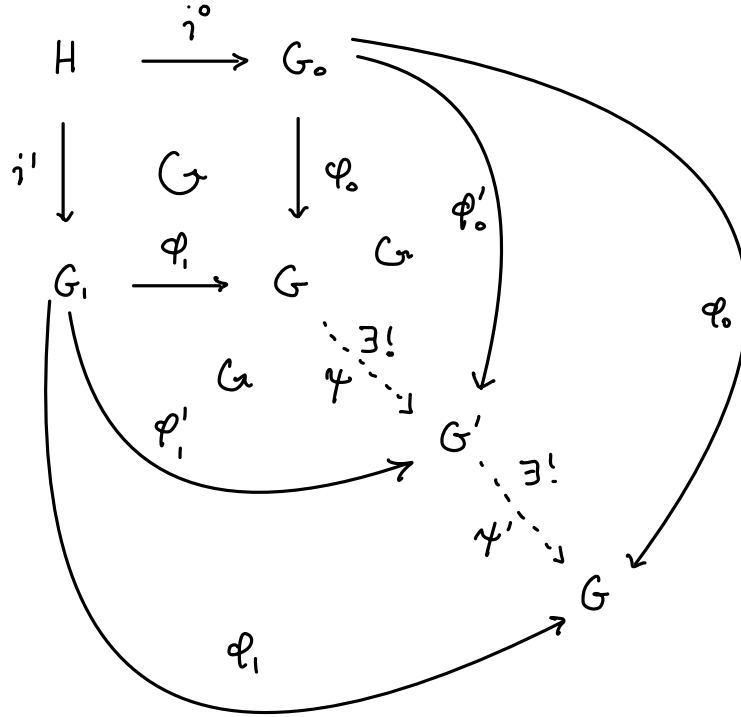
Fact:

A map $\bar{\varphi}: G \rightarrow H$ st $N \subseteq \text{Ker}(\bar{\varphi})$ descends to a well-defined map $\varphi: G/N \rightarrow H$ by $\varphi([g]) = \bar{\varphi}(g)$.

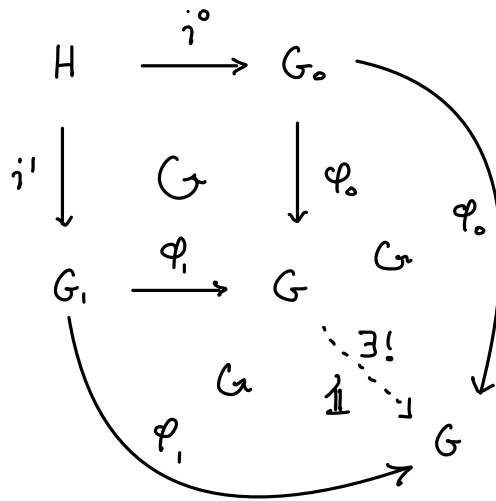
Lemma:

Such a G is unique.

Proof: Let G' be another such group.



Note



So by uniqueness, $\psi' \circ \psi = \mathbb{1} \Rightarrow G \cong G'$ by a unique isom \square

Defn: $G = G_0 *_{\mathbb{H}} G_1$

Defn: $G, H = \text{grps}$, The free product of two groups G, H is the group

$$G * H = G *_0 H = \langle S_G \cup S_H \mid R_G \cup R_H \rangle$$

Ex: $\underbrace{\mathbb{Z} * \dots * \mathbb{Z}}_{n\text{-times}} = F_n = \text{free group on } n \text{ letters.}$