Lecture #17 - November 9th, 2023

Lemma: $f, g : X \to Y$ are homotopic rel $x_0 \in X$, then $f_* = g_*$.

Proof: Let $H_t : X \to Y$ be the homotopy from f to g.

$$f_*([\alpha]) = [f \circ \alpha] = [H_0 \circ \alpha] = [H_1 \circ \alpha] = [g \circ \alpha] = g_*([\alpha]) \qquad \square$$

Prop: If $f : X \to Y$ is a homotopy equiv, then $f_* =$ isom.

Lemma: $H_t : X \to Y$ is a hpty and $\gamma(s) = H_s(x_0) : I \to Y$, then

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[r, "(H_1)_*"] \arrow[d, "(H_0)_*"'] & \pi_1(Y, H_1(x_1)) \arrow[dl, "\Phi_\gamma"] \\ \pi_1(Y, H_0(x_0)) & \end{tikzcd}$$

↳ ie, $(H_0)_* = \Phi_\gamma \circ (H_1)_*$

Proof: NTS $\forall [\alpha] \in \pi_1(X, x_0)$, $\gamma \cdot (H_1 \circ \alpha) \cdot \gamma^{-1} \simeq H_0 \circ \alpha$ rel ∂I.

$$G_t(s) = \begin{cases} \gamma(3st) & , \ 0 \le s \le 1/3 \\ H_t \circ \alpha(3s-1) & , \ 1/3 \le s \le 2/3 \\ \gamma(t(3-3s)) & , \ 2/3 \le s \le 1 \end{cases}$$

G_t cts by Pasting Lemma.

$G_t(0) = H_0(x_0) = G_t(1)$, $G_0 \simeq H_0 \circ \alpha$, $G_1 \simeq \gamma \cdot H_1 \circ \alpha \cdot \gamma^{-1}$

$\Rightarrow H_0 \circ \alpha \simeq \gamma \cdot H_1 \circ \alpha \cdot \gamma^{-1}$ rel ∂I $\qquad \square$

Proof: Let $g: Y \to X$ be st $f \circ g \simeq \mathbb{1}_Y$, $g \circ f \simeq \mathbb{1}_X$.

Let H_t be hpty from $f \circ g$ to $\mathbb{1}_Y$. $\gamma(s) = H_s(x_0)$

" G_t " " " $g \circ f$ to $\mathbb{1}_X$. $\delta(s) = G_s(f(x_0))$.

$$\left.\begin{array}{l} f_* \circ g_* = \Phi_\gamma \\ g_* \circ f_* = \Phi_\delta \end{array}\right\} \Rightarrow \begin{array}{l} f_* \circ g_* \circ \Phi_\gamma^{-1} = \mathbb{1} \Rightarrow f_* \text{ surj} \\ \Phi_\delta^{-1} \circ g_* \circ f_* = \mathbb{1} \Rightarrow f_* \text{ inj} \end{array}$$

$\square$

$\pi_1(S^1) = \mathbb{Z}$

Notn: $\pi: \mathbb{R} \to S^1$, $\pi(\lambda) = (\cos(2\pi \cdot \lambda), \sin(2\pi \cdot \lambda))$

Fact: ① Given $\gamma: I \to S^1$ w/ $\gamma(0) = (1,0) \in S^1$ and $n \in \mathbb{Z}$, $\exists!$ cts map $\tilde{\gamma}: I \to \mathbb{R}$ st $\pi \circ \tilde{\gamma} = \gamma$ and $\tilde{\gamma}(0) = n$

② Given a hpty $H_t: I \to S^1$ w/ $H_t(0) = (1,0) \in S^1$ and $n \in \mathbb{Z}$, $\exists!$ cts map $\tilde{H}_t: I \to \mathbb{R}$ st $\pi \circ \tilde{H}_t = H_t$ and $\tilde{H}_t(0) = n$

Thm: $\pi_1(S^1) = \mathbb{Z}$.

Proof: Step 1: Define a map $\Phi: \pi_1(S^1) \to \mathbb{Z}$.

If $\alpha(0) = (1,0) = \alpha(1) = (1,0)$, then $\tilde{\alpha}(1) \in \mathbb{Z}$.

So defn $\Phi: \pi_1(S^1) \to \mathbb{Z}$ by $\Phi([\alpha]) = \tilde{\alpha}(1)$.

NTS Φ is well-defn

If $\alpha \simeq \beta$ rel $\{0,1\}$ w/ hpty H_t

By Fact ②, $\exists\ \tilde{H}_t: I \to \mathbb{R}$ w/ $\pi \circ \tilde{H}_t = H_t$, $\tilde{H}_t(0) = 0$.

By uniqueness in Fact ①, $\tilde{\alpha} = \tilde{H}_0$ and $\tilde{\beta} = \tilde{H}_1$.

$\tilde{H}_t(1)$ is cts and $\tilde{H}_t(1) \in \mathbb{Z} \Rightarrow \tilde{H}_t(1) =$ constant.

$\Rightarrow \tilde{\alpha}(1) = \tilde{H}_0(1) = \tilde{H}_1(1) = \tilde{\beta}(1)$.

$\Rightarrow \Phi$ is well-defined.

Step 2: Φ is a homomorphism

Spse $[\alpha], [\beta] \in \pi_1(S^1)$

Defn $\tilde{\gamma} : I \longrightarrow \mathbb{R}$ by

$$\tilde{\gamma}(s) = \begin{cases} \tilde{\alpha}(2s) & , \quad 0 \le s \le 1/2 \\ \tilde{\alpha}(1) + \tilde{\beta}(2s-1) & , \quad 1/2 \le s \le 1 \end{cases}$$

So $\tilde{\gamma}$ is a lift of $\alpha \cdot \beta$ w/ $\tilde{\gamma}(0) = 0$

$\Phi([\alpha]\cdot[\beta]) = \Phi([\alpha \cdot \beta]) = \tilde{\gamma}(1) = \tilde{\alpha}(1) + \tilde{\beta}(1)$

Step 3: Φ is surjective

$\tilde{\gamma}_n : I \to \mathbb{R}$, $\tilde{\gamma}_n(t) = n \cdot t$.

Defn $\gamma_n = \pi \circ \tilde{\gamma}_n$, so $\tilde{\gamma}_n$ = lift of γ_n based at 0.

$\Rightarrow \Phi([\gamma_n]) = \tilde{\gamma}_n(1) = n \in \mathbb{Z}$

Step 4: Φ is injective

$\Phi([\alpha]) = 0 \Rightarrow \tilde{\alpha}(1) = 0$

Defn $\tilde{H}_t : I \to \mathbb{R}$ by $\tilde{H}_t(s) = t \cdot \tilde{\alpha}(s)$.

Defn $H_t = \pi \circ \tilde{H}_t$.

$H_0 = \pi \circ \tilde{H}_0 = (1,0)$

$H_1 = \pi \circ \tilde{H}_1 = \pi \circ \tilde{\alpha} = \alpha$

$H_t(0) = \pi \circ \tilde{H}_t(0) = \pi \circ (t \cdot \tilde{\alpha}(0)) = \pi(0) = (1,0)$

$H_t(1) = \pi \circ \tilde{H}_t(1) = \pi \circ (t \cdot \tilde{\alpha}(1)) = \pi(0) = (1,0)$

$\Rightarrow [\alpha] = [e]$

$\square$

Cor: $\mathbb{R}^2$ is not homeo to $\mathbb{R}^n$ for $n \neq 2$.

Proof: $\pi_1(S^{n-1}) = 0$ for $n > 2$, $\pi_1(S^1) = \mathbb{Z}$

$\Rightarrow S^{n-1} \not\simeq S^1$ for $n \neq 1$

$\Rightarrow S^1 \simeq \mathbb{R}^2 \setminus 0 \not\simeq \mathbb{R}^n \setminus 0 \simeq S^{n-1}$ for $n \neq 1$

$\Rightarrow \mathbb{R}^2 \setminus 0 \not\cong \mathbb{R}^n \setminus 0$ for $n \neq 1$

$\Rightarrow \mathbb{R}^2 \cong \mathbb{R}^n$ for $n \neq 1$.

$\square$

Brouwer's Fixed Point Thm

Lemma: If X retracts onto A, then $i: A \hookrightarrow X$ gives an inj hom $i_*: \pi_1(A) \to \pi_1(X)$. If X defo retracts onto A, then $i_* =$ isom.

Proof: Let $r: X \to A$ be the retraction.

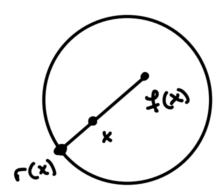
$r \circ i = \mathbb{1} \Rightarrow r_* \circ i_* = (r \circ i)_* = \mathbb{1}_* \Rightarrow i_* =$ injective.

If X defo retracts $\Rightarrow i$ is a hpty equiv. $\Rightarrow i_* =$ isom. $\square$

Thm: (Brouwer's Fixed Pt theorem) $f: D^2 \to D^2$ cts $\Rightarrow \exists\, x$ st $f(x) = x$.

Proof: Spse that $f(x) \neq x \ \forall x \in X$.

Defn $r: D^2 \to S^1$ by



$r =$ retract $\Rightarrow i_*: \pi_1(S^1) = \mathbb{Z} \hookrightarrow \pi_1(D^2) = 0$ is inj. $\Rightarrow\Leftarrow$ $\square$