Lecture #16 - November 7^{th}, 2023

Notn: $\Omega_{x_0} X = \{ \alpha : I \to X \mid \alpha(0) = x_0 = \alpha(1) \}$ for $x_0 \in X$.

Define an equiv. rel. on $\Omega_{x_0} X$ by $\alpha \sim \beta$ iff $\alpha \simeq \beta$ rel $\{0,1\}$.

Defn: Given $\alpha, \beta \in \Omega_{x_0} X$, define $\alpha \cdot \beta \in \Omega_{x_0} X$ by

$$\alpha \cdot \beta(s) = \begin{cases} \alpha(2s), & 0 \le s \le 1/2 \\ \beta(2s-1), & 1/2 \le s \le 1 \end{cases}$$

$\hookrightarrow$ $\alpha \cdot \beta(0) = \alpha(0) = x_0 = \beta(1) = \alpha \cdot \beta(1)$

Defn: The <u>fundamental group</u> of X based at x_0 is the set

$$\pi_1(X, x_0) = \Omega_{x_0} X / \sim$$

w/ group law $[\alpha] \cdot [\beta] = [\alpha \cdot \beta]$

Proof: (Well-defn) NTS $\alpha \sim \alpha'$ and $\beta \sim \beta'$ $\Rightarrow$ $[\alpha \cdot \beta] = [\alpha' \cdot \beta']$

$\alpha \sim \alpha'$ $\Rightarrow$ $\alpha \simeq \alpha'$ rel $\{0,1\}$ w/ hmpty say H_t.

$\beta \sim \beta'$ $\Rightarrow$ $\beta \simeq \beta'$ " " " " " G_t.

Define

$$F_t(s) = \begin{cases} H_t(2s), & 0 \le s \le 1/2 \\ G_t(2s-1), & 1/2 \le s \le 1 \end{cases}$$

$H_t(1) = x_0 = G_t(0)$ $\Rightarrow$ F_t = cts by Pasting Lemma.

$F_0(s) = \alpha \cdot \beta(s)$, $F_1(s) = \alpha' \cdot \beta'$, $F_t(0) = H_t(0) = x_0 = G_t(1) = F_t(1)$

$\Rightarrow$ $\alpha \cdot \beta \simeq \alpha' \cdot \beta'$ rel $\{0,1\}$.

$\Rightarrow$ $[\alpha \cdot \beta] = [\alpha' \cdot \beta']$

(Unit) $e : I \to X$, $e(s) = x_0$. NTS $[e]\cdot[\alpha] = [\alpha]$

$$e\cdot\alpha = \begin{cases} x_0, & 0 \le s \le 1/2 \\ \alpha(2s-1), & 1/2 \le s \le 1 \end{cases}$$

$e\cdot\alpha$ = reparam of $\alpha \Rightarrow e\cdot\alpha \simeq \alpha$ rel $\{0,1\}$

$\Rightarrow [e]\cdot[\alpha] = [e\cdot\alpha] = [\alpha]$

Similarly, $[\alpha]\cdot[e] = [\alpha]$.

(Associative):

$$\alpha\cdot(\beta\cdot\gamma) = \begin{cases} \alpha(2s) & 0 \le s \le 1/2 \\ \beta(4s-2) & 1/2 \le s \le 3/4 \\ \gamma(4s-3) & 3/4 \le s \le 1 \end{cases}$$

$$(\alpha\cdot\beta)\cdot\gamma = \begin{cases} \alpha(4s), & 0 \le s \le 1/4 \\ \beta(4s-1), & 1/4 \le s \le 1/2 \\ \gamma(2s-1), & 1/2 \le s \le 1 \end{cases}$$

Note, $\alpha\cdot(\beta\cdot\gamma)$ = reparam of $(\alpha\cdot\beta)\cdot\gamma$

$\Rightarrow \alpha\cdot(\beta\cdot\gamma) \simeq (\alpha\cdot\beta)\cdot\gamma$ rel $\{0,1\}$

$$\begin{aligned} \Rightarrow [\alpha]\cdot([\beta]\cdot[\gamma]) &= [\alpha]\cdot[\beta\cdot\gamma] \\ &= [\alpha\cdot(\beta\cdot\gamma)] \\ &= [(\alpha\cdot\beta)\cdot\gamma] \\ &= [\alpha\cdot\beta]\cdot[\gamma] \\ &= ([\alpha]\cdot[\beta])\cdot[\gamma] \end{aligned}$$

(Inverses): $\alpha : I \to X$, define $\alpha^{-1} : I \to X$, $\alpha^{-1}(s) = \alpha(1-s)$.

$$\alpha\cdot\alpha^{-1}(s) = \begin{cases} \alpha(2s), & 0 \le s \le 1/2 \\ \alpha^{-1}(2s-1), & 1/2 \le s \le 1 \end{cases} = \begin{cases} \alpha(2s), & 0 \le s \le 1/2 \\ \alpha(2-2s), & 1/2 \le s \le 1 \end{cases}$$

$$H_t(s) = \begin{cases} \alpha(2s), & 0 \le s \le t/2 \\ \alpha(2t-2s), & t/2 \le s \le t \\ e(s), & t \le s \le 1 \end{cases}$$

$\alpha(2(t/2)) = \alpha(t) = \alpha(2t - 2(t/2))$, $\alpha(2t-2t) = e(s)$

$\Rightarrow H_t$ cts by Pasting Lemma.

$H_0(s) = e(s)$, $H_1(s) = \alpha \cdot \alpha^{-1}(s)$, $H_t(0) = \alpha(0) = x_0 = e(s) = H_t(1)$

$\Rightarrow \alpha \cdot \alpha^{-1} \simeq e$ rel $\{0,1\}$

$\Rightarrow [\alpha] \cdot [\alpha^{-1}] = [\alpha \cdot \alpha^{-1}] = [e]$

Sim. $[\alpha^{-1}] \cdot [\alpha] = [e]$

So $[\alpha^{-1}] = [\alpha]^{-1}$ $\square$

Cor: $\alpha : I \to X$, $\beta : I \to X$ st

① $\alpha \simeq \alpha'$ rel ∂I

② $\beta \simeq \beta'$ rel ∂I

③ $\alpha(1) = \beta(0)$

$\Rightarrow \alpha \cdot \beta \simeq \alpha' \cdot \beta'$ rel ∂I

Change of basepoint

Prop: Let $\gamma : I \to X$ be a path from x_0 to x_1. Define

$\Phi_\gamma : \pi_1(X, x_1) \longrightarrow \pi_1(X, x_0)$, $\Phi_\gamma([\alpha]) = [\gamma \cdot (\alpha \cdot \gamma^{-1})]$

Φ_γ is a grp isomorphism.

Proof: (Well-defn): NTS $[\alpha] = [\beta] \Rightarrow \Phi_\gamma([\alpha]) = \Phi_\gamma([\beta])$

$[\alpha] = [\beta] \Rightarrow \alpha \simeq \beta$ rel ∂I

By Cor, $\alpha \cdot \gamma^{-1} \simeq \beta \cdot \gamma^{-1}$ rel $\partial I \Rightarrow \gamma \cdot (\alpha \cdot \gamma^{-1}) \simeq \gamma \cdot (\beta \cdot \gamma^{-1})$ rel ∂I.

$\Rightarrow \Phi_\gamma([\alpha]) = \Phi_\gamma([\beta])$

(Hom): $\Phi_\gamma([\alpha]\cdot[\beta]) = \Phi_\gamma([\alpha\cdot\beta])$

$$
\begin{aligned}
&= [\gamma\cdot((\alpha\cdot\beta)\cdot\gamma^{-1})] && \text{reparam} \\
&= [\gamma\cdot(((\alpha\cdot e)\cdot\beta)\cdot\gamma^{-1})] && \text{cor} \\
&= [\gamma\cdot(((\alpha\cdot(\gamma^{-1}\cdot\gamma))\cdot\beta)\cdot\gamma^{-1})] && \text{reparam} \\
&= [(\gamma\cdot(\alpha\cdot\gamma^{-1}))\cdot(\gamma\cdot(\beta\cdot\gamma^{-1}))] \\
&= \Phi_\gamma([\alpha])\cdot\Phi_\gamma([\beta])
\end{aligned}
$$

(Inverse): $\Phi_{\gamma^{-1}}\circ\Phi_\gamma([\alpha]) = \Phi_{\gamma^{-1}}([\gamma\cdot(\alpha\cdot\gamma^{-1})])$

$$
\begin{aligned}
&= [\gamma^{-1}\cdot((\gamma\cdot(\alpha\cdot\gamma^{-1}))\cdot\gamma)] && \text{reparam} \\
&= [(\gamma^{-1}\cdot\gamma)\cdot(\alpha\cdot(\gamma^{-1}\cdot\gamma))] && \text{cor} \\
&= [e\cdot(\alpha\cdot e)] && \text{reparam} \\
&= [\alpha]
\end{aligned}
$$

Similarly, $\Phi_\gamma\circ\Phi_{\gamma^{-1}} = \mathbb{1}$ □

Simply-connected spaces

Defn: X = simply-connected iff X = path-conn and $\pi_1(X, x_0) = 0$.

Prop: X = simply-conn iff $\exists!$ hpty class of paths connecting any 2 pts in X.

Proof: $(\Rightarrow)$: Spse $f, g : I \to X$ connect $x_0, x_1 \in X$

$f\cdot g^{-1}$, $g^{-1}\cdot g \in \pi_1(X) = 0 \Rightarrow$ homotopic to constant maps

By Cor. $\Rightarrow f \simeq f\cdot g^{-1}\cdot g \simeq g$ rel ∂I

$(\Leftarrow)$: $[e], [\alpha] \in \pi_1(X) \Rightarrow e, \alpha$ conn. x_0 to x_1 in X

$\Rightarrow e \simeq \alpha$ rel $\partial I \Rightarrow [\alpha] = [e] \Rightarrow \pi_1(X) = 0$ □

Prop: $\pi_1(S^n) = 0$ for $n \geq 2$.

Warning: $\exists$ surj cts maps $\gamma : S^1 \longrightarrow S^n$ $\forall n \geq 0$!

Proof: Let $H_i^{\pm} = \{ x \in S^n \mid \pm x_i > 0 \}$ = open cover of S^n.

Let $[\alpha] \in \pi_1(S^n)$.

$\alpha^{-1}(H_i^{\pm})$ cover I.

Leb. covering lem, $\exists \frac{1}{N} > 0$ st $\forall s \in I$, $B_{\frac{1}{N}}(s) \subseteq \alpha^{-1}(H_i^{\pm})$ for some $i, \pm$.

Write $[0,1] = [0,\frac{1}{N}] \cup [\frac{1}{N}, \frac{2}{N}] \cup \dots \cup [\frac{N-1}{N}, 1]$.

$\forall i$, $\alpha([\frac{i-1}{N}, \frac{i}{N}]) \subseteq H_j^{\pm} \cong \mathbb{R}^n$ for some $j, \pm$.

$\Rightarrow \alpha|_{[\frac{i-1}{N}, \frac{i}{N}]} \simeq$ great circle path from $\alpha(\frac{i-1}{N})$ to $\alpha(\frac{i}{N})$ rel $\partial[\frac{i-1}{N}, \frac{i}{N}]$

$\Rightarrow \alpha \simeq$ concatenation of pieces of great circles rel ∂I.

$\Rightarrow \alpha \simeq \beta$ rel ∂I st $\beta : I \to S^n$ is <u>not</u> surj.

$\Rightarrow \beta : I \longrightarrow S^n \setminus pt \cong \mathbb{R}^n \Rightarrow \beta \simeq e$ rel ∂I.

$\Rightarrow [\alpha] = [\beta] = [e] \Rightarrow \pi_1(S^n) = 0$ $\square$

Induced Homomorphisms

Defn: $f : X \to Y$ gives a homomorphism

$$f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, f(x_0))$$

by $f_*([\alpha]) = [f \circ \alpha]$.

Proof: (Well-defn): Spse $[\alpha] = [\beta] \Rightarrow \alpha \simeq \beta$ rel $\{0,1\}$ by a hpty H_t.

Defn $G_t = f \circ H_t$.

$G_0 = f \circ \alpha$, $G_1 = f \circ \beta$

$G_t(0) = f \circ H_t(0) = f(x_0)$, $G_t(1) = f \circ H_t(1) = f(x_0)$

$\Rightarrow f \circ \alpha \simeq f \circ \beta$ rel $\partial I \Rightarrow f_*([\alpha]) = f_*([\beta])$

$$\begin{aligned}(\text{Hom}): f_*([\alpha]\cdot[\beta]) &= f_*([\alpha\cdot\beta]) \\ &= [f\circ(\alpha\cdot\beta)] \\ &= [(f\circ\alpha)\cdot(f\circ\beta)] \\ &= [f\circ\alpha]\cdot[f\circ\beta] \\ &= f_*([\alpha])\cdot f_*([\beta])\end{aligned}$$

$\square$

Cor: ① $f: X \to Y$, $g: Y \to Z$, $g_* \circ f_* = (g\circ f)_*$

② $\mathbb{1}: X \to X$ is the identity map, $\mathbb{1}_* =$ identity map.

Cor: $f: X \to Y$ is a homeo $\Rightarrow \pi_1(X) \cong \pi_1(Y)$