Lecture # 15 - November 1^{st}, 2023

Isomorphisms

Defn: A hom $\varphi: G \to G'$ is an <u>isomorphism</u> iff it is bijective.

↳ We say G is isomorphic to G'; write $G \cong G'$.

Ex: $\exp: (\mathbb{R}, +) \longrightarrow (\mathbb{R}_{>0}, \times)$ is an isomorphism.

Lemma: $\varphi: G \to G' = \text{isom} \Rightarrow \varphi^{-1} = \text{isom}$.

Proof: NTS φ^{-1} is a hom.

Spse $a', b' \in G'$, $\exists!\, a, b \in G$ st $\varphi(a) = a'$, $\varphi(b) = b'$

$$\varphi^{-1}(a' \cdot b') = \varphi^{-1}(\varphi(a) \cdot \varphi(b)) = \varphi^{-1}(\varphi(a \cdot b)) = a \cdot b = \varphi^{-1}(a') \cdot \varphi^{-1}(b') \quad \square$$

Normal Subgroups and Quotients

Defn: A subgroup $N \subseteq G$ is <u>normal</u> if $\forall n \in N, \forall g \in G$, $g^{-1} \cdot n \cdot g \in N$.

Ex: $\text{Ker}(\varphi) \subseteq G$ is normal

$n \in \text{Ker}(\varphi)$, $g \in G \Rightarrow \varphi(g^{-1} \cdot n \cdot g) = \varphi(g)^{-1} \cdot \varphi(n) \cdot \varphi(g) = \varphi(g)^{-1} \cdot \varphi(g) = e'$.

Ex: $SL_n(\mathbb{R}) \subseteq GL_n(\mathbb{R})$ is a normal subgroup

Ex: Every subgroup of an abelian group is normal.

Lemma: $H \subseteq G$ determines an equivalence relation by $g \sim h$ iff $gh^{-1} \in H$

Proof: $gg^{-1} = e \in H \Rightarrow g \sim g$

$g \sim h \Rightarrow gh^{-1} \in H \Rightarrow hg^{-1} = (gh^{-1})^{-1} \in H \Rightarrow h \sim g$

$gh^{-1}, hK^{-1} \in H \Rightarrow gK^{-1} = gh^{-1}hK^{-1} \in H \Rightarrow (g \sim h,\ h \sim K \Rightarrow g \sim K)$ $\square$

Lemma: $N \subseteq G$ normal, G/N = equiv classes is a group w/

$$[g]\cdot[h] = [g\cdot h]$$

Proof: NTS $\cdot$ is well-defined: $g \sim g',\ h \sim h' \Rightarrow [g\cdot h] = [g'\cdot h']$.

$h \sim h' \Rightarrow hh'^{-1} \in N \Rightarrow hh'^{-1} = n \in N$

$gh\cdot(g'\cdot h')^{-1} = gh\cdot h'^{-1}\cdot g'^{-1} = g\cdot n\cdot g'^{-1} \in N$ by normal!

$\Rightarrow gh \sim g'h' \Rightarrow [g\cdot h] = [g'\cdot h']$.

① $[e]$ = unit: $[e]\cdot[g] = [e\cdot g] = [g] = [g\cdot e] = [g]\cdot[e]$.

② $([g]\cdot[h])\cdot[K] = [g\cdot h]\cdot[K] = [g\cdot h\cdot K] = [g]\cdot[h\cdot K] = [g]\cdot([h]\cdot[K])$

③ $[g^{-1}]\cdot[g] = [g^{-1}\cdot g] = [e]$ $\square$

Defn: $N \subseteq G$ normal, G/N = <u>quotient</u> of G by N.

Lemma: $q: G \to G/N,\ g \mapsto [g]$ is a surj. hom w/ $\mathrm{Ker}(q) = N$.

Proof: By defn, q is surjective.

$g \in N \iff g \sim e \iff [g] = [e] \iff g \in \mathrm{Ker}(q)$ $\square$

Thm: Let $\varphi: G \to H$ be a surjective group hom. $H \cong G/\mathrm{Ker}(\varphi)$.

Proof: Define $\Phi: G/\mathrm{Ker}(\varphi) \longrightarrow H$ by $\Phi([g]) = \varphi(g)$.

If $g \sim h \Rightarrow gh^{-1} \in \mathrm{Ker}(\varphi) \Rightarrow \Phi([g]) = \varphi(g) = \varphi(h) = \Phi([h])$.

$\Rightarrow \Phi$ is well-defined.

$$\Phi([g]\cdot[h]) = \Phi([g\cdot h]) = \varphi(gh) = \varphi(g)\cdot\varphi(h) = \Phi([g])\cdot\Phi([h])$$

$\Rightarrow \Phi = \text{hom}$

By construction, Φ is surjective.

Spse $\Phi([g]) = 0 \Rightarrow \varphi(g) = 0 \Rightarrow g \in \text{Ker}(\varphi) \Rightarrow g \sim e \Rightarrow [g] = [e]$.

$\Rightarrow \Phi$ is injective. $\square$

Ex: $GL_n(\mathbb{R})/SL_n(\mathbb{R}) \cong (\mathbb{R}\setminus 0, \times)$

det : $GL_n(\mathbb{R}) \longrightarrow \mathbb{R} \setminus 0$ is surjective.

$\text{Ker}(\det) = SL_n(\mathbb{R})$.

Defn: The <u>order</u> of a group G is $|G|$.

<u>Order of an element</u> $g \in G$ is the minimal # n st $g^n = e$.

Lemma: If order of $g \in G$ is n, then $\langle g \rangle \cong \mathbb{Z}/n$

Proof: Defn $\varphi : \mathbb{Z} \longrightarrow \langle g \rangle \subseteq G$ by $\varphi(k) = g^k$.

φ is surjective.

$\varphi(k) = e \Rightarrow g^k = e \Rightarrow k = \ell \cdot n \Rightarrow k \in n\cdot\mathbb{Z}$.

$\Rightarrow \langle g \rangle = \mathbb{Z}/n\cdot\mathbb{Z}$ $\square$

Fundamental Groups

Rem: A cts map $\alpha : I \to X$ st $\alpha(0) = \alpha(1)$ is equiv. to a map $\bar{\alpha} : S^1 \to X$ st

$$\begin{tikzcd} I \arrow[r, "\alpha"] \arrow[d, "q"'] & X \\ S^1 \arrow[ur, "\bar{\alpha}"'] & \end{tikzcd}$$

w/ $\bar{\alpha}([x]) = \alpha(x)$. $\bar{\alpha} = $ cts since $\bar{\alpha} \circ q = \alpha$ is cts.

Sim. $\alpha \simeq \beta$ rel $\{0,1\}$ iff $\bar{\alpha} \simeq \bar{\beta}$ rel $\{[0]\}$.

↳ We will not distinguish the difference between α and $\bar{\alpha}$.

Notn: $\Omega_{x_0} X = \{ \alpha : I \to X \mid \alpha(0) = x_0 = \alpha(1) \}$ for $x_0 \in X$.

Define an equiv. rel. on $\Omega_{x_0} X$ by $\alpha \sim \beta$ iff $\alpha \simeq \beta$ rel $\{0,1\}$.

Proof: $(\alpha \sim \alpha)$: $H_t : I \to X$ by $H_t(s) = \alpha(s)$.

$(\alpha \sim \beta \Rightarrow \beta \sim \alpha)$: Spse $H_t : I \to X$ st $H_0(s) = \alpha(s)$, $H_1(s) = \beta(s)$

Defn $G_t : I \to X$ by $G_t(s) = H_{1-t}(s)$.

$\Rightarrow$ $G_0(s) = H_1(s) = \beta(s)$, $G_1(s) = H_0(s) = \alpha(s)$

$G_t(0) = H_{1-t}(0) = x_0 = H_{1-t}(1) = G_t(1)$

$\Rightarrow$ $\beta \simeq \alpha$ rel $\{0,1\}$ $\Rightarrow$ $\beta \sim \alpha$

$(\alpha \sim \beta, \beta \sim \gamma \Rightarrow \alpha \sim \gamma)$: Spse $H_t : I \to X$ st $H_0(s) = \alpha(s)$, $H_1(s) = \beta(s)$

" $G_t : I \to X$ st $G_0(s) = \beta(s)$, $G_1(s) = \gamma(s)$

$$F_t(s) = \begin{cases} H_{2t}(s), & 0 \le t \le 1/2 \\ G_{2t-1}(s), & 1/2 \le t \le 1 \end{cases}$$

By Pasting Lemma, F is cts

Note, $F_0(s) = H_0(s) = \alpha(s)$, $F_1(s) = G_1(s) = \gamma(s)$,

$$F_t(0) = \begin{cases} H_{2t}(0) \\ G_{2t-1}(0) \end{cases} = x_0 = \begin{cases} H_{2t}(1) \\ G_{2t-1}(1) \end{cases} = F_t(1)$$

$\Rightarrow \alpha \simeq \gamma \text{ rel } \{0,1\} \Rightarrow \alpha \sim \gamma$ $\square$

Defn: Given $\alpha, \beta \in \Omega_{x_0} X$, define $\alpha \cdot \beta \in \Omega_{x_0} X$ by

$$\alpha \cdot \beta(s) = \begin{cases} \alpha(2s), & 0 \le s \le 1/2 \\ \beta(2s-1), & 1/2 \le s \le 1 \end{cases}$$

$\hookrightarrow \alpha \cdot \beta(0) = \alpha(0) = x_0 = \beta(1) = \alpha \cdot \beta(1)$

Defn: The <u>fundamental group</u> of X based at x_0 is the set

$$\pi_1(X, x_0) = \Omega_{x_0} X / \sim$$

w/ group law $[\alpha] \cdot [\beta] = [\alpha \cdot \beta]$

Lemma: $\pi_1(\mathbb{R}^n, 0) = 0$ = trivial group w/ 1 element.

Proof: STS $\forall \alpha \in \Omega_0 \mathbb{R}^n$, $\alpha \sim e$.

$H_t(s) = t \cdot \alpha(s)$.

$H_0(s) = 0 = e(s)$, $H_1(s) = \alpha(s)$

$H_t(0) = t \cdot \alpha(0) = 0 = t \cdot \alpha(1) = H_t(1)$

$\Rightarrow \alpha \sim e$ $\square$