Lecture #13 - October 24th, 2023

Defn: A CW-complex is a space built in the following manner:

① Start w/ discrete set X^0 whose pts are called 0-cells.

② Let $D^n_\alpha = n$-ball, $S^{n-1}_\alpha = \partial D^n_\alpha$.

Inductively, X^n is the quotient of $X^{n-1} \sqcup_\alpha D^n_\alpha$ by rel.

$x \sim \varphi_\alpha(x)$ for $x \in \partial D^n_\alpha$ where $\varphi_\alpha : \partial D^n_\alpha \longrightarrow X^{n-1}$.

↳ $X^n = X^{n-1} \cup_\alpha e^n_\alpha$ w/ $e^n_\alpha = \text{int}(D^n_\alpha) = n$-cells

↳ φ_α are called the attaching maps

③ If one does not stop at a finite n, then $X = \bigcup_n X^n$

Weak topology: $U \subseteq X$ open iff $U \cap X_n$ open $\forall n$.

The characteristic map of a cell e^n_α is the map

$$\Phi_\alpha : D^n_\alpha \hookrightarrow X^{n-1} \sqcup_\beta D^n_\beta \xrightarrow[\text{quot.}]{} X^n \longrightarrow X$$

Ex: Γ = Graph = CW-cpx

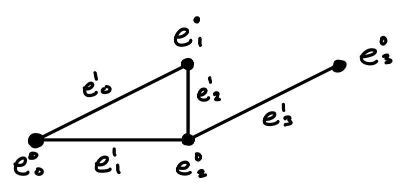
$\Gamma^0 = \bigsqcup_{\alpha \in V(\Gamma)} e^0_\alpha$

Given $\beta \in E(\Gamma)$ define $\varphi_\beta : \partial(D^1_\beta) = \{0,1\} \longrightarrow \Gamma^0$ by

$\varphi_\beta(0)$ = tail of edge β and $\varphi_\beta(1)$ = head of edge β

$\Gamma^1 = \Gamma^0 \cup_\beta e^1_\beta = \Gamma$

↳ Eg:

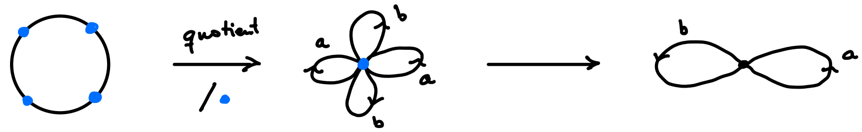


Ex: Torus = CW-cpx:

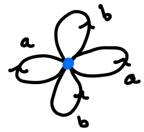
$X^0 = e^0_v$ = 0-cell

$X^1 = X^0 \sqcup e^1_a \cup e^1_b$ w/ $\left.\begin{array}{l} \varphi_a : \partial D^1_a = \{0,1\} \to X^0 \\ \varphi_b : \partial D^1_b = \{0,1\} \to X^0 \end{array}\right\} =$ contant maps

$X^2 = X^1 \sqcup e^2$ w/ $\varphi : \partial D^2 = S^1 \to X^1$ by



↳ Gluing w/



gives a Klein bottle. Attaching maps matter!

Ex: S^n = CW-cpx; $X^0 = e^0$, $X^k = e^0$ for $k < n$, $X^n = pt \sqcup e^n$ w/ $\varphi : \partial D^n \to pt$ via constant map.

$\Rightarrow X^n = D^n / \partial D^n = S^n$

Ex: $\mathbb{RP}^n = S^n/(x \sim -x) = D^n/(x \sim -x \text{ w/ } x \in \partial D^n)$

$\varphi : \partial D^n = S^{n-1} \longrightarrow \mathbb{RP}^{n-1}$ via the quotient yields an attaching map

$\Rightarrow \mathbb{RP}^0 \subseteq \mathbb{RP}^1 \subseteq \mathbb{RP}^2 \subseteq \dots \subseteq \mathbb{RP}^{n-1} \subseteq \mathbb{RP}^n$

The characteristic map is the quotient map $D^n \longrightarrow \mathbb{RP}^n$

Inductively, $\mathbb{RP}^n$ has CW-structure w/ cells $e^0 \cup e^1 \cup e^2 \cup \dots \cup e^n$.

Continuing this process, we get $\mathbb{RP}^\infty = \bigcup_n \mathbb{RP}^n$.

Defn: A subcomplex of a CW-cpx X is a closed subspace $A \subseteq X$ that is the union of cells of X.

The pair (X, A) is called a CW-pair

Lemma: $A \subseteq X$ a subcpx $\Rightarrow A$ = CW-cpx

Proof: Spse $e^n_\alpha \subseteq A$ is an n-cell. A = closed

$\Phi_\alpha : D^n_\alpha \longrightarrow X$ cts + A = closed

$\Rightarrow \Phi_\alpha(\text{int}(D^n_\alpha)) = e^n_\alpha \subseteq A \Rightarrow \Phi_\alpha(D^n_\alpha) \subseteq \overline{e^n_\alpha} \subseteq A$

$\Rightarrow \Phi_\alpha$ has image in A.

$\Rightarrow A$ is obtained via a seq. of cell-attachments. $\square$

Ex: $\mathbb{RP}^k \subseteq \mathbb{RP}^n$ is a subcomplex.

Ex: $S^k \subseteq S^n$ is <u>not</u> a subcpx w/ our cell structure, but it can be a subcpx for a different CW-str. on S^n.

$\hookrightarrow$ $S^0 = e^0_S \cup e^0_N$

$$S^1 = S^0 \cup (e^1_S \cup e^1_N) \quad \text{w/} \quad \left.\begin{array}{l} \varphi^1_S : \partial D^1 \to S^0 \\ \varphi^1_N : \partial D \to S^0 \end{array}\right\} = \text{identity map}$$

$$\vdots$$

$$S^n = S^{n-1} \cup (e^n_S \cup e^n_N) \quad \text{w/} \quad \left.\begin{array}{l} \varphi^n_S : \partial D^n \to S^{n-1} \\ \varphi^n_N : \partial D^n \to S^{n-1} \end{array}\right\} = \quad \vdots$$

Inductively, we get $S^\infty = \bigcup_n S^n$

Also <u>now</u> $S^k \subseteq S^n$ is a subcomplex.

Rem: ① $X, Y =$ CW-cpxes $\Rightarrow$ $X \times Y$ has structure of CW-cpx.

e^n_α, e^m_β are cells in X, Y $\Rightarrow$ $e^n_\alpha \times e^m_\beta$ is a cell in $X \times Y$ w/ characteristic map given by the product:

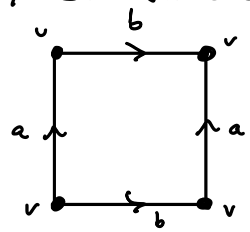
$$\Phi_{\alpha,\beta} : D^{n+m} \cong D^n \times D^m \longrightarrow X \times Y, \quad \Phi_{\alpha,\beta} = \Phi_\alpha \times \Phi_\beta.$$

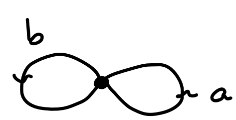
② $(X, A) =$ CW-pair $\Rightarrow$ $X/A =$ CW-cpx

Cells of X/A are cells in $X \setminus A$ and one new 0-cell = image of A in X/A.

Given a cell e^n_α of $X \setminus A$ w/ attaching map $\varphi_\alpha : \partial D^n \to X^{n-1}$, we get an attaching map $\tilde{\varphi}_\alpha : \partial D^n \to X^{n-1} \longrightarrow X^{n-1}/A^{n-1}$

Ex: Torus w/ CW-structure

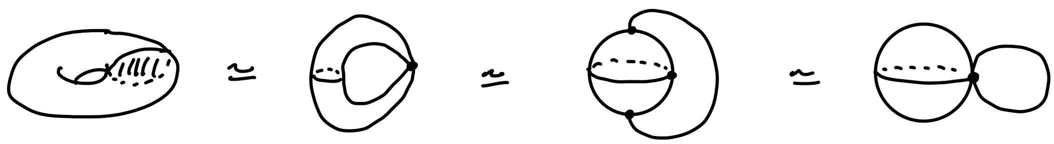
X 

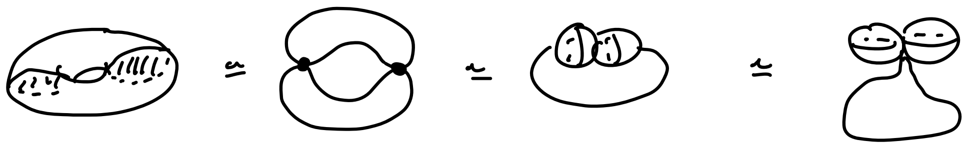
, $A =$ 

, $X/A = S^2$

Prop: If $(A, X) =$ CW-pair w/ A contractible, then the quotient $q : X \to X/A$ is a homotopy equivalence.

Ex:

① 

② 

Defn: $A \subseteq X$ satisfies the <u>homotopy</u> <u>extension</u> <u>property</u> if $\forall$ $f : X \to Y$, $H_t : A \to Y$ w/ $H_0 = f|_A$, $\exists$ homotopy $f_t : X \to Y$ st $f_t|_A = H_t$.

Lemma: $(X, A) =$ CW pair $\Rightarrow$ $A \subseteq X$ satisfies HEP.

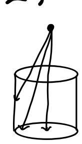
Rem: $H_t : X \to Y$, $G_t : X \to Y$ are homotopies st $H_1 = G_0$, then we can concatenate $H * G$ to get a new hpty

$$(H * G)_t(x) = \begin{cases} H_{2t}(x) & , \quad 0 \le t \le 1/2 \\ G_{2t-1}(x) & , \quad 1/2 \le t \le 1 \end{cases}$$

cts by Pasting Lemma.

Proof: Step 1: $D^n \times 0 \cup \partial D^n \times I$ is a defo retract of $D^n \times I$

Pf: $\exists$ retraction $r : D^n \times I \to D^n \times 0 \cup \partial D^n \times I \subseteq D^n \times I$ via radial projection from $(0, 2) \in D^n \times \mathbb{R} \subseteq \mathbb{R}^{n+1}$



Defn $H_t : D^n \times I \to D^n \times I$, $H_t = t \cdot r + (1-t) \cdot \mathbb{1}$.

$H_t =$ defo retract. $\square$

Step 2: $X^n \times \{0\} \cup (X^{n-1} \cup A^n) \times I \subseteq X^n \times I$ is a defo retract of $X^n \times I$

Pf: Apply the above deformation retract simultaneously on every n-cell of X^n that is not in A^n; Call it H_t^n. $\square$

Step 3: Concatenate the H_t^n's to get defo retract of $X \times I$ onto $X \times 0 \cup A \times I$.

Pf: Spse $X = X^n$ for simplicity. (one can concatenate a countable # of homotopies, which one does for the general result.)

$$\begin{tikzcd}
X \times I \arrow[r, "H^n"] \arrow[rdddd, bend right, "r"'] & X \times 0 \cup (X^{n-1} \cup A) \times I \arrow[d, "H^{n-1}"] \\
& X \times 0 \cup ((X^{n-1} \times 0 \cup (X^{n-2} \cup A^{n-1})) \cup A) \times I \arrow[d, equal] \\
& X \times 0 \cup (X^{n-2} \cup A) \times I \arrow[d, "H^{n-2}"] \\
& \vdots \arrow[d] \\
& X \times 0 \cup A \times I
\end{tikzcd}$$

$\square$

Step 4: Spse $f : X \to Y$, $H_t : A \to Y$ as in HEP defn.

Define $f_t(x) = H \circ r(x,t)$

$f_0(x) = H \circ r(x, 0) = H_0(x) = x$

$x \in A$, $f_t(x) = H \circ r(x,t) = H(x,t) = H_t(x)$. $\square$

<u>Cor:</u> $A \subseteq X$ satisfies the HEP if $X \times 0 \cup A \times I$ is a retract of $X \times I$.

<u>Proof of Prop:</u> Let $H_t : A \to X$ be contraction of A w/ H_0 = inclusion of A.

HEP $\Rightarrow \exists\, f_t : X \to X$ w/ $f_0 = \mathbb{1}$, $f_t|_A = H_t$

$\exists$ maps $\bar{f}_t : X/A \to X/A$, $g : X/A \to X$ st

$$\begin{tikzcd}
X \arrow[r, "f_t"] \arrow[d, "q"'] \arrow[dr, phantom, "\circlearrowleft"] & X \arrow[d, "q"] \\
X/A \arrow[r, "\bar{f}_t"'] & X/A
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "f_1"] \arrow[d, "q"'] \arrow[dr, phantom, "\circlearrowleft"] & X \\
X/A \arrow[ur, "g"'] &
\end{tikzcd}$$

$$g \circ q = f_1 \simeq f_0 = \mathbb{1}_X$$

$$q \circ g([x]) = q \circ g \circ q(x) = q \circ f_1(x) = \bar{f}_1(q(x)) = \bar{f}_1([x])$$

$$\Rightarrow \quad q \circ g = \bar{f}_1 \simeq \bar{f}_0 = \mathbb{1}_{X/A}.$$

$\Rightarrow$ q is homotopy equiv. $\square$

Lemma: Every CW-cpx is normal.

Ex: $A, B \subseteq X$ be closed + disjoint.

Construct $f_n : X^n \longrightarrow I$ st

① $f_n(A \cap X^n) = 0$

② $f_n(B \cap X^n) = 1$

③ $f_m = f_n|_{X^m}$ for $m \leq n$.

f_0 : $X^0 = A \cap X^0 \sqcup Y \sqcup B \cap X^0$ are collections of disj pts

$f_0(A \cap X^0) = 0$, $f_0(Y) = 1/2$, $f_0(B \cap X^0) = 1$.

f_n : Spse we have constructed f_{n-1}

Given a cell e_α^n w/ attaching map $\varphi_\alpha : \partial D_\alpha^n \longrightarrow X^{n-1}$

and char. map $\Phi_\alpha : D_\alpha^n \longrightarrow X^n$.

$f_{n-1} \circ \varphi_\alpha : \partial D_\alpha^n \longrightarrow I$

Define $\tilde{g}_\alpha : \partial D_\alpha^n \cup \Phi_\alpha^{-1}(A) \cup \Phi_\alpha^{-1}(B) \longrightarrow I$ by

$$\tilde{g}_\alpha(x) = \begin{cases} f_{n-1} \circ \varphi_\alpha(x), & x \in \partial D_\alpha^n \\ 0, & x \in \Phi_\alpha^{-1}(A) \\ 1, & x \in \Phi_\alpha^{-1}(B) \end{cases}$$

Tietz $\Rightarrow$ $\tilde{g}_\alpha$ extends to map $g_\alpha : D_\alpha^n \longrightarrow I$ st

① $g_\alpha(\Phi_\alpha^{-1}(A)) = 0$

② $g_\alpha(\Phi_\alpha^{-1}(B)) = 0$

③ $g_\alpha|_{\partial D_\alpha^n} = f_{n-1} \circ \varphi_\alpha$

Pasting lemma $\Rightarrow$ g_α is continuous.

Doing this for all n-cells e_α^n, we may use the pasting lemma to glue f_{n-1} w/ the g_α to obtain f_n.

Define $f : X \to I$ by $f(x) = f_n(x)$ if $x \in X^n$.

f cts iff $f^{-1}(U)$ open iff $f^{-1}(U) \cap X^n = f_n^{-1}(U) =$ open $\forall n$ iff f_n cts