Lecture #12 - October 24th, 2023

Thm: Let X = cpt metric space w/ covering dimension $\leq n$. There exists a cts injective map $f : X \longrightarrow \mathbb{R}^{2n+1}$.

Thm: X = manifold iff X = embedded manifold.

Lemma: X = space, $f : X \longrightarrow \mathbb{R}^N$ w/ $f^{-1}(\text{cpt}) = \text{cpt}$, then f is closed

Proof: $C \subseteq X$ closed. Spse $y \in \mathbb{R}^N \setminus f(C)$

HB, $\overline{B_y(\varepsilon)}$ = cpt

By hypo, $f^{-1}(\overline{B_y(\varepsilon)})$ = compact

$K = C \cap f^{-1}(\overline{B_y(\varepsilon)})$ = compact (closed of cpt = cpt)

$\Rightarrow f(K)$ = cpt = closed ($\mathbb{R}^N$ = Haus) and $f(K) \subseteq f(C)$

$y \in V = B_y(\varepsilon) \setminus f(K)$ = open.

$z \in f(C) \cap V \Rightarrow \exists\ x \in f^{-1}(B_y(\varepsilon)) \cap C \subseteq K$ st $f(x) = z$

$\Rightarrow z \in f(K)$

$\Rightarrow z \notin V$

$\Rightarrow y \in V \subseteq \mathbb{R}^N \setminus f(C)$

$\Rightarrow f(C)$ = closed $\square$

Proof of Thm: X = manifold. By HW, $\exists\ f : X \to \mathbb{R}_{\geq 0}$ st $f^{-1}(\text{cpt}) = \text{cpt}$

cpt $= K_i = f^{-1}([0,i]) \subseteq f^{-1}([0,i+1)) \subseteq \text{int}(K_{i+1})$

Note $X = \bigcup_i K_i$

cpt $= K_i \setminus \text{int}(K_{i-1}) = C_i \subseteq \mathcal{U}_i = \text{int}(K_{i+1}) \setminus K_{i-2}$ = open

X = normal + Urysohn $\Rightarrow \exists$ cts $\rho_i : X \to \mathbb{R}$ w/

① $\rho_i(C_i) = 1$

② $\rho_i(X \setminus \mathcal{U}_i) = 0$.

By Thm, $\exists$ cts, closed, inj $f_i : D_i =: K_{i+1} \setminus int(K_{i-2}) \longrightarrow \mathbb{R}^{2n+1}$

Define $\psi_i : X \longrightarrow \mathbb{R}^{2n+1}$ by

$$\psi_i(x) = \begin{cases} \rho_i(x) \cdot f_i(x) & , \; x \in U_i \\ 0 & , \; \text{else} \end{cases}$$

Defn $\psi : X \to \mathbb{R}^{4n+3}$,

$$\psi(x) = \left(\sum_{i=ev} \psi_i(x) , \sum_{i=odd} \psi_i(x) , f(x) \right)$$

Step 1: ψ = inj.

↳ Spse $\psi(x) = \psi(y)$

$\Rightarrow f(x) = f(y)$

$\Rightarrow x, y \in C_i$ WLOG i = even

$\Rightarrow \psi_{i \neq 2k}(x) = 0 \;\; \forall K.$

$\Rightarrow \sum_{j=ev} \psi_j(x) = \psi_i(x) = f_i(x)$

$\Rightarrow \psi(x) = (f_i(x), \text{stuff}) = (f_i(y), \text{stuff}) = \psi(y)$

$\Rightarrow x = y$

Step 2: ψ = closed

↳ Spse $K \subseteq \mathbb{R}^N$ cpt $\Rightarrow K$ = closed

closed $= \psi^{-1}(K) \subseteq f^{-1}($proj of K to last $\mathbb{R}$ = cpt$)$ = cpt

$\Rightarrow \psi^{-1}(K)$ = cpt.

Above Lemma completes the proof □

Homotopy

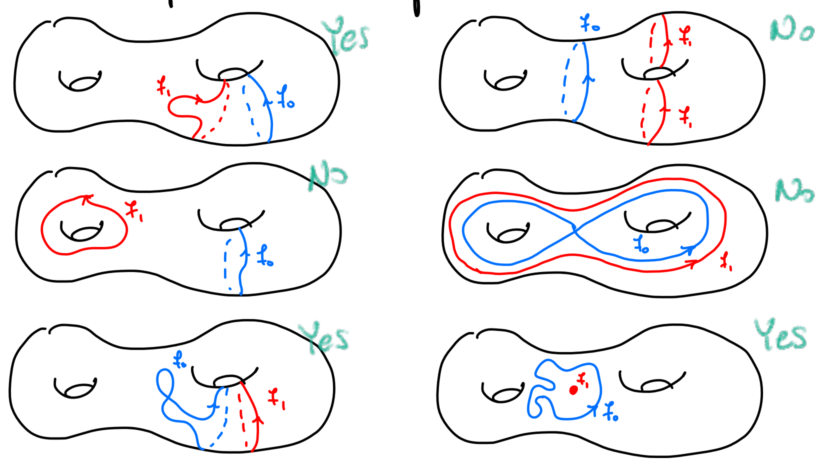
Defn: Given $f_0, f_1 : X \longrightarrow Y$, a <u>homotopy</u> from f_0 to f_1 is a map $H : X \times I \to Y$ st $f_0(x) = H(x, 0)$ and $f_1(x) = H(x, 1) \;\; \forall x.$

↳ Sometimes write $H(x,t) = H_t(x)$

One says that f_0, f_1 are homotopic

↳ One writes $f_0 \simeq f_1$

Ex: Which maps are homotopic?



Defn: A homotopy rel $A \subseteq X$ is a homotopy $H: X \times I \to Y$ st $H_t(x)$ is ind. of t $\forall x \in A$.

Defn: A <u>reparameterization</u> of $\alpha: I \to X$ is a map $\beta: I \to X$ st $\beta = \alpha \circ r$ where $r: I \to I$ satisfies $r(0) = 0$, $r(1) = 1$.

Lemma: β = reparam of $\alpha \Rightarrow \alpha \simeq \beta$ rel $\{0,1\}$

Proof: $H_t(s) = \alpha\big(t \cdot r(s) + (1-t)\cdot s\big)$.

Note H is cts since it is prod/sum/comp of cts functions

$H_0(s) = \alpha(s)$, $H_1(s) = \alpha \circ r(s) = \beta$.

$H_t(0) = \alpha(t \cdot r(0) + (1-t)\cdot 0) = \alpha(0)$

$H_t(1) = \alpha(t \cdot r(1) + (1-t)\cdot 1) = \alpha(1)$ $\square$

Defn: A map $f: X \to Y$ is a homotopy equivalence if $\exists$ a map $g: Y \to X$ st $f \circ g \simeq \mathbb{1}_Y$, $g \circ f \simeq \mathbb{1}_X$.

↪ X, Y are said to be homotopy equivalent.

↪ Write $X \simeq Y$.

Warning: $X \cong Y \Rightarrow X \simeq Y$ but $X \simeq Y \not\Rightarrow X \cong Y$.

Ex: Which letters are homotopy equivalent: A, B, C, D, O, P, Q, S?

Ex: $\mathbb{R}^n \simeq pt$.

$f: \mathbb{R}^n \to pt$, $f(x) = 0$

$g: pt \to \mathbb{R}^n$, $g(pt) = 0$

$f \circ g = \mathbb{1}_{pt}$

$H_t(x) = t \cdot x$, $H_0(x) = g \circ f(x)$, $H_1(x) = x \Rightarrow H_t$ hpty $g \circ f \simeq \mathbb{1}_{\mathbb{R}^n}$

Defn: ① X is contractible if $X \simeq pt$

② $f: X \to Y$ is <u>null-homotopic</u> if $f \simeq$ constant map.

Defn: A retraction of X onto $A \subseteq X$ is a cts map $r: X \to X$ st $r|_A = \mathbb{1}_A$, $r(X) = A$.

Defn: A <u>deformation retraction</u> is a homotopy rel. A $H: X \times I \to X$ st $H_0 = \mathbb{1}_X$ and $H_1 =$ retraction onto A.

↪ Say X deformation retracts onto A

Lemma: X defo retracts onto $A \Rightarrow X \simeq A$.

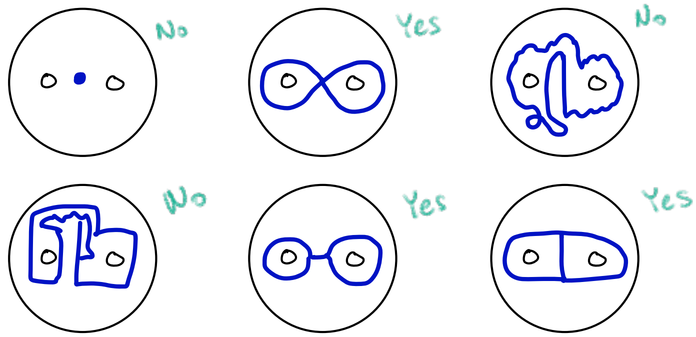
Proof: Let $r: X \to r(X) = A \subseteq X$ be the retract

Let $i: A \to X$ be the inclusion.

$r \circ i = \mathbb{1}_A$ by defn

Let $H_t =$ hpty above. $H_0 = \mathbb{1}_X$, $H_1 = r = i \circ r \Rightarrow i \circ r \simeq \mathbb{1}_X$. □

Ex: Which spaces defo retract onto the blue?



Ex: $S^1 \times I$ deformation retracts onto $S^1 \times \{0\}$

$\hookrightarrow$ $H : (S^1 \times I) \times I \longrightarrow S^1 \times I$ by $H_t(\theta, s) = (\theta, s \cdot (1-t))$

$H_0(\theta, s) = (\theta, s)$, $H_1(\theta, s) = (\theta, 0)$, $H_t(\theta, 0) = (\theta, 0)$

$\Downarrow$ $\quad\quad$ $\Downarrow$ $\quad\quad$ $\Downarrow$

$H_0 = \mathbb{1}$ $\quad\quad$ $H_1 =$ retract $\quad\quad$ $H_t =$ hpty rel. $S^1 \times \{0\}$.

Ex: The Möbius band defo retracts onto its core circle