Lecture # 11 - October 19th, 2023

Defn: X = space. The covering dimension of X is infimum over $n \in \mathbb{N}$ st $\forall\, \mathcal{U}_\alpha$ = open cover $\exists$ refinement V_β st $\forall x \in X$, x meets at most $n+1$ of the V_β.

Lemma: The covering dimension of $[0,1]^n$ is bdd by $2^n - 1$

Fact: The covering dim of $[0,1]^n$ is n

Exer: $X \cong Y \Rightarrow$ covering dimension of X = covering dimension of Y.

Lemma: $X = A \cup B$ w/ A, B = closed, then $\dim(X) = \max(\dim(A), \dim(B))$

Proof: Let $\mathcal{U}_\alpha$ = cover of X. $\exists$ refinement V_β st $A \cap V_\beta$ cover A and $x \in A$ meets at most $\dim(A)+1$ of the $A \cap V_\beta$.

Consider $V_\beta = \{ V_\beta, \mathcal{U}_\alpha \setminus A \}$

$\exists$ refinement W_γ st $B \cap W_\gamma$ cover B, $x \in B$ meets at most $\dim(B)+1$ of the $B \cap W_\gamma$.

$\forall \gamma$ pick β_γ st $W_\gamma \subseteq V_{\beta_\gamma}$

Define $Y_\beta = \bigcup_{\beta_\gamma = \beta} W_\gamma \subseteq V_\beta$ (each W_γ is in a unique Y_β)

Note, Y_β = cover + refinement

If $x \in B$, then x meets $\dim(B)+1$ many W_γ's

$\Rightarrow$ x meets $\dim(B)+1$ many Y_β.

If $x \in A$, then x meets $\dim(A)+1$ many V_β's.

But $Y_\beta \subseteq V_\beta \Rightarrow x$ " " " Y_β's. $\square$

Exer: $A \subseteq X$ is closed and the covering dim of X is $\leq n$, then the covering dim of A is $\leq n$.

Cor: Every cpt subspace of an n-manifold has covering dim $\leq n$

Thm: X = cpt metric space of covering dim n. $\exists$ cts, inj map $f: X \to \mathbb{R}^{2n+1}$.

Lemma 1: (Baire's Thm) X = cpt + Haus, $\mathcal{U}_n$ = countable collection of dense opens. $\Rightarrow \bigcap_n \mathcal{U}_n$ = dense.

Defn: X = cpt metric space

① $C(X, \mathbb{R}^{2n+1}) = \{ f : X \to \mathbb{R}^{2n+1} \mid f = \text{cts} \}$.

Metric on $C^0(X, \mathbb{R}^{2n+1})$ via $d(f, g) = \sup_x |f(x) - g(x)|$

↳ X = cpt $\Rightarrow d$ is well-defn metric.

② $A \subseteq X$ $\operatorname{diam}(A) = \sup_{x, y \in A} d(x, y)$

③ $f \in C(X, \mathbb{R}^{2n+1})$, $\operatorname{diam}(f) = \sup \{ \operatorname{diam}(f^{-1}(z)) \mid z \in f(X) \}$

④ $\mathcal{U}_\varepsilon = \{ f \in C(X, \mathbb{R}^{2n+1}) \mid \operatorname{diam}(f) < \varepsilon \}$

Remark: $\bigcap_n \mathcal{U}_{1/n}$ = injective maps in $C(X, \mathbb{R}^{2n+1})$

Lemma 2: $\mathcal{U}_\varepsilon$ is open

Lemma 3: $\mathcal{U}_\varepsilon$ is dense

Proof: (of Thm) $\bigcap_n \mathcal{U}_{1/n} \neq \emptyset \Rightarrow \exists$ cts, inj. $f: X \to \mathbb{R}^{2n+1}$.

X = cpt, $f(X)$ = Haus $\Rightarrow f$ = closed $\square$

Proof 1: NTS $\bigcap_n \mathcal{U}_n$ meets every open in X. Let W = open.

$\mathcal{U}_1$ dense + open $\Rightarrow \exists\ x_1 \in \mathcal{U}_1 \cap W$ = open

Normal $\Rightarrow \exists\ x_2 \in W_2 \subseteq \overline{W_2} \subseteq \mathcal{U}_1 \cap W$.

Inductively, $\exists\ x_n \in \mathcal{U}_n \cap W_n$ = open $\Rightarrow \exists\ x_{n+1} \in W_{n+1} \subseteq \overline{W_{n+1}} \subseteq W_n \cap \mathcal{U}_n$.

Note, $\overline{W_{n+1}} \subseteq \overline{W_n} \subseteq \dots$. So X = cpt $\Rightarrow \bigcap_n \overline{W_n} \neq \phi$

Note, $\overline{W_n} \subseteq \mathcal{U}_k \cap W \quad \forall k > n$

$\Rightarrow \bigcap_n \overline{W_n} \subseteq \bigcap_k \mathcal{U}_k \cap W = W \cap \bigcap_k \mathcal{U}_k$

$\Rightarrow W \cap \bigcap_n \mathcal{U}_n \neq \phi$. $\square$

Proof 2: $f \in \mathcal{U}_\varepsilon$, NTS $\exists\ \delta > 0$ st $d(f,g) < \delta \Rightarrow g \in \mathcal{U}_\varepsilon$.

$A_f = \{ (x,y) \in X \times X \mid d(x,y) \geq \text{diam}(f) + \varkappa \}$ = closed = cpt

where $\text{diam}(f) + \varkappa < \varepsilon$.

Since $d(x,y) > \text{diam}(f) \Rightarrow f(x) \neq f(y)$ for $(x,y) \in A_f$.

$\Rightarrow |f(x) - f(y)| > 0$ on A_f.

Let $\delta = (\text{min. of } |f(x) - f(y)| \text{ on } A_f)/2$

Spse $d(f,g) < \delta$.

$$\Rightarrow |g(x) - g(y)| = |g(x) - f(x) + f(x) + f(y) - f(y) - g(y)|$$
$$\geq -|g(x) - f(x)| + |f(x) - f(y)| - |f(y) - g(y)|$$
$$> 0 \quad \text{on } A_f.$$

$\Rightarrow$ If $g(x) = g(y) \Rightarrow (x,y) \notin A_f \Rightarrow d(x,y) < \text{diam}(f) + \varkappa < \varepsilon$

$\Rightarrow \text{diam}(g) < \varepsilon$

$\Rightarrow g \in \mathcal{U}_\varepsilon$ $\square$

Defn: $\{z_0, \dots, z_m\} \subseteq \mathbb{R}^n$ are geom. ind iff

$\sum_i \lambda_i \cdot z_i = 0$ and $\sum_i \lambda_i = 0 \Rightarrow \lambda_i = 0$

$\hookrightarrow \Leftrightarrow \left(-\lambda_0 = \sum_{i=1}^n \lambda_i,\ 0 = \sum_{i=0}^n \lambda_i \cdot z_i = \sum_{i=1}^n \lambda_i \cdot (z_i - z_0) \right) \Rightarrow \lambda_i = 0$

$\Leftrightarrow \sum_{i=1}^n \lambda_i \cdot (z_i - z_0) = 0 \Rightarrow \lambda_i = 0$

$\Leftrightarrow \{z_i - z_0\}$ are lin. ind.

Defn: $\{z_0, \dots, z_m\} \subseteq \mathbb{R}^n$ are in general position iff any subset w/ $\leq n+1$ elms are geom. ind.

Lemma 4: Given $\{z_0, \dots, z_m\} \subseteq \mathbb{R}^n$ and $\delta > 0$, $\exists \{y_0, \dots, y_m\} \subseteq \mathbb{R}^n$ st

① $\{y_i\}$ in general position

② $|z_i - y_i| < \delta \ \forall i$.

Proof: $\mathcal{G} : \prod_{\{i_0, \dots, i_n\} \subseteq \{0, \dots, m\}} \det(x_{i_1} - x_{i_0}, \dots, x_{i_n} - x_{i_0}) : (\mathbb{R}^n)^{m+1} \longrightarrow \mathbb{R}$

$\mathcal{G}((x_0, \dots, x_m)) \neq 0$ iff $(x_0, \dots, x_m)$ in gen. pos.

$\mathcal{G}^{-1}(\mathbb{R} \setminus 0) = \text{open} + \text{dense}$ Fact!

$\Rightarrow B_z(\varepsilon) \cap \mathcal{G}^{-1}(\mathbb{R} - 0) \neq \emptyset$ w/ $\varepsilon \ll \delta$.

$\Rightarrow \exists (y_0, \dots, y_m)$ w/ $|z_i - y_i| < \delta$ and in gen. pos. $\square$

Proof 3: Given $f \in C(X, \mathbb{R}^{2n+1})$, NTS $\forall \delta > 0 \ \exists g \in \mathcal{U}_\varepsilon$ w/ $d(f,g) < \delta$.

Fix cover of X by $\mathcal{U}_1, \dots, \mathcal{U}_m$

① $\operatorname{diam}(\mathcal{U}_i) < \varepsilon/2$

② $\operatorname{diam}(f(\mathcal{U}_i)) < \delta/2$

③ Each $x \in X$ meets at most $n+1$ of the $\mathcal{U}_i$ $(\dim(X) \leq n)$.

$\hookrightarrow$ Leb covering Lem $\Rightarrow \exists\, 0 < \varkappa < \frac{\varepsilon}{4}$ st $B_x(\varkappa) \subseteq f^{-1}(B_y(\delta/4))$ for some y

$\exists$ refinement $\mathcal{U}_\beta$ of $B_x(\varkappa)$ st ③ holds.

$\mathcal{U}_\beta \subseteq B_{x_\beta}(\varkappa)$ for some $x_\beta \Rightarrow$ ① + ② hold

$X = \text{cpt} \Rightarrow \exists$ finite # $\mathcal{U}_1, \dots, \mathcal{U}_m$ of $\mathcal{U}_\alpha$ that cover.

Let $\phi_i : X \to \mathbb{R}$ be a partition of unity ass. to the $\mathcal{U}_i$.

Fix $x_i \in \mathcal{U}_i$ and $z_i \in \mathbb{R}^{2n+1}$ st

ⓐ $d(f(x_i), z_i) < \delta/2$

ⓑ $\{z_1, \dots, z_m\}$ are in general position

Defn $g : X \to \mathbb{R}^{2n+1}$ by $g(x) = \sum_i \phi_i(x) \cdot z_i$.

Claim: $d(f,g) < \delta$

$\hookrightarrow \left| g(x) - f(x) \right| = \left| \sum_i \phi_i(x) \cdot z_i - \sum_i \phi_i(x) \cdot f(x) \right|$

$\quad = \left| \sum_i \phi_i(x) \cdot (z_i - f(x_i)) + \sum_i \phi_i(x) \cdot (f(x_i) - f(x)) \right|$

ⓐ $\quad \leq \sum_i \phi_i(x) \cdot \delta/2 + \sum_i \phi_i(x) \cdot \left| f(x_i) - f(x) \right|$

part. of unity + ② $\quad \leq \sum_i \phi_i(x) \cdot \delta/2 + \sum_i \phi_i(x) \cdot \delta/2$

$\quad = \delta$ $\square$

Claim: $g \in \mathcal{U}_\varepsilon$.

$\hookrightarrow$ If $g(x) = g(y) \Rightarrow \sum_i (\phi_i(x) - \phi_i(y)) \cdot z_i = 0$

x, y meet at most $n+1$ of the $\mathcal{U}_i$

$\Rightarrow$ Only $2n+2$ of the above terms are non-zero.

Note, $\sum_i (\phi_i(x) - \phi_i(y)) = 1 - 1 = 0$.

z_i in gen pos $\Rightarrow$ Any subset w/ $\leq 2n+2$ elm are geom. ind.

$\Rightarrow \phi_i(x) = \phi_i(y) \quad \forall i$

Note $\phi_i(x) > 0$ for some i

$\Rightarrow x, y \in \mathcal{U}_i \Rightarrow d(x,y) < \varepsilon/2 \Rightarrow \operatorname{diam}(g) \leq \varepsilon/2 < \varepsilon$ $\square$