Lecture #10 - October 17th, 2023

Manifolds

Defn: A n-dim'l manifold X is a 2^{nd} countable Hausdorff space st $\forall x \in X$, $\exists$ open $x \in \mathcal{U}$ and a homeo $\varphi_x : \mathbb{R}^n \longrightarrow \mathcal{U}$.

Defn: An embedded n-manifold is a subspace $X \subseteq \mathbb{R}^N$ st $\forall x \in X$ $\exists$ open $x \in \mathcal{U}$ and a homeo $\varphi : \mathbb{R}^n \longrightarrow \mathcal{U}$.

Thm: X = manifold iff X = embedded manifold.

Proof: (M = cpt manifold $\Rightarrow$ M = emb manifold)

$\exists$ $\mathcal{U}_1, \dots, \mathcal{U}_N$ st

① $\varphi_i : \mathbb{R}^n \longrightarrow \mathcal{U}_i$ homeo

② $\varphi_i(B(1))$ cover M.

Normal + Urysohn $\Rightarrow$ $\exists$ $\rho_i : M \to \mathbb{R}$ st

① $\rho_i(\varphi_i(\overline{B(1)})) \equiv 1$

② $\rho_i(M \setminus \varphi_i(B(2))) \equiv 0$

Defn $\psi_i : M \to \mathbb{R}^{n+1}$,

$$\psi_i(x) = \begin{cases} (\rho_i(x), \rho_i(x) \cdot \varphi_i^{-1}(x)) & , x \in \mathcal{U}_i \\ 0 & , \text{else} \end{cases}$$

Defn $\psi : M \to \mathbb{R}^{n \cdot N + N}$, $\psi(x) = (\dots, \psi_i(x), \dots)$.

Defn: X = space is <u>paracompact</u> iff it is Haus and every open cover admits a locally finite refinement.

Exer: $A \subseteq X$ = paracpt w/ A = closed $\Rightarrow$ A = paracpt.

Lemma: X = paracpt $\Rightarrow$ X = normal

Proof: Fix A, B = closed + disj.

Step 1: $\forall x \in X \setminus B$, $\exists$ opens st $x \in \mathcal{U}$, $B \subseteq V$, $\mathcal{U} \cap V = \emptyset$.

↳ Pf: $\forall y \in B$ $\exists$ $\mathcal{U}_y \ni x$, $V_y \ni y$ st $\mathcal{U}_y \cap V_y = \emptyset$ (X = Haus)

$\Rightarrow (\bigcup_y V_y) \cup (X \setminus B)$ cover X

paracpt $\Rightarrow$ $\exists$ locally finite refinement V_α that cover B

$\exists$ open $W \ni x$ st W meets only finitely many V_α's.

Say W meets $V_1, \dots, V_n \subseteq V_{y_1}, \dots, V_{y_n}$.

Define $\mathcal{U} = W \cap (\bigcap_i \mathcal{U}_{y_i})$, $V = \bigcup_\alpha V_\alpha$.

Note, $x \in \mathcal{U}$ = open, $B \subseteq V$ = open.

If $z \in \mathcal{U}$ $\Rightarrow$ $z \in \mathcal{U}_{y_i}$ $\Rightarrow$ $z \notin \bigcup_{i=1}^n V_i$

$\Rightarrow$ $z \in W$ $\Rightarrow$ $z \notin V \setminus \bigcup_{i=1}^n V_i$

(together) $\Rightarrow z \notin V$

$\Rightarrow \mathcal{U} \cap V = \emptyset$.

Step 2: Prove normal.

↳ Pf: $\forall x \in A$, $\exists$ $\mathcal{U}_x, V_x$ opens st $x \in \mathcal{U}_x$, $B \subseteq V_x$, $\mathcal{U}_x \cap V_x = \emptyset$.

$\Rightarrow (\bigcup_x \mathcal{U}_x) \cup (X \setminus A)$ cover X

paracpt $\Rightarrow$ $\exists$ locally finite refinement $\mathcal{U}_\alpha$ that cover A.

Define $\mathcal{U} = \bigcup_\alpha \mathcal{U}_\alpha$

$\forall y \in B$, $\exists$ W_y st W_y meets finitely many $\mathcal{U}_\alpha$

Spse W_y meets $\mathcal{U}_1, \dots, \mathcal{U}_n \subseteq \mathcal{U}_{x_1}, \dots, \mathcal{U}_{x_n}$

Define $V_y = W_y \cap (\bigcap_{i=1}^n V_{x_i})$

Note, $y \in V_y$, $V_y \cap \mathcal{U} = \emptyset$.

Define $V = \bigcup_{y \in B} V_y \supseteq B$.

$\mathcal{U}, V$ give the desired separating neighborhoods. $\square$

Defn: A <u>partition of unity</u> on X is an open cover $\bigcup_\alpha U_\alpha$ that is locally finite and $\exists\ \rho_\alpha : X \to I$ st

① $\rho_\alpha(x) > 0 \Rightarrow x \in U_\alpha$

② $\Sigma_\alpha \rho_\alpha(x) = 1$

Lemma: Every cover of a paracpt space admits a refinement that has a partition of unity.

Proof: Let U_α = cover.

X = paracpt $\Rightarrow$ WLOG U_α locally finite.

Normal $\Rightarrow \forall\ x \in U_{\alpha_x}$, $\exists\ x \in W_x \subseteq \overline{W_x} \subseteq U_{\alpha_x}$

Let V_β = locally finite refinement of W_x's.

So $V_\beta \subseteq W_{x_\beta} \subseteq \overline{W}_{x_\beta} \subseteq U_{\alpha_{x_\beta}}$.

X = normal + Urysohn $\Rightarrow \psi_\beta : X \to I$ st

① $\psi_\beta(\overline{V}_\beta) \equiv 1$,

② $\psi_\beta(X \setminus U_{\alpha_{x_\beta}}) = 0$

Define $\rho_\beta(x) = \psi_\beta(x) / \Sigma_\gamma \psi_\gamma(x)$

V_β cover $\Rightarrow$ denominator is non-zero.

Given $x \in X$, $\exists\ U \ni x$ st U meets finitely many V_β's.

$\Rightarrow \rho_\beta(x)$ is well-defined + cts

Set $U_\beta = U_{\alpha_{x_\beta}}$

① $\rho_\beta(x) > 0 \Rightarrow x \in U_\beta$.

② $\Sigma_\beta \rho_\beta(x) = 1$ (use locally finite as above)

Prop: X = manifold $\Rightarrow X$ = paracpt.

Lemma: X = manifold, $\exists\ K_n$ = cpt w/ $K_i \subseteq \text{int}(K_{i+1})$, $X = \bigcup_n \text{int}(K_n)$

Proof: Let $\mathcal{U}_1, \dots, \mathcal{U}_n, \dots$ be st $\varphi_i : \mathbb{R}^n \to \mathcal{U}_i$ homeo, $\varphi_i(B(1))$ cover

$K_{i,m} := \varphi_i(\overline{B_0(m)}) = \text{cpt}$, $\text{int}(K_{i,m}) \supseteq \varphi_i(B_0(m))$

$K_n := \bigcup_{i,m \le n} K_{i,m} = \text{cpt}$, $\text{int}(K_n) \supseteq \bigcup_{i,m \le n-1} \varphi_i(\overline{B_0(m)}) = K_{n-1}$

Note $K_n \subseteq K_{n+1}$ and $X = \bigcup_i \mathcal{U}_i = \bigcup_i \bigcup_m \varphi_i(B_0(m)) \subseteq \bigcup_n \text{int}(K_n)$ □

Exer: X = manifold. $\exists$ cts $f : X \to \mathbb{R}$ st $f^{-1}(\text{cpt}) = \text{cpt}$.

Proof: Let $\bigcup_\alpha \mathcal{U}_\alpha = X$ be an open cover.

$K_n = \text{cpt} \Rightarrow \exists\ \mathcal{U}^n_{i_1}, \dots, \mathcal{U}^n_{i_n}$ that cover K_n.

$V^n_{i_j} := \mathcal{U}^n_{i_j} \cap (X \setminus K_{n-1}) \subseteq \mathcal{U}^n_{i_j}$ (Note K_{n-1} = closed since X = Haus)

Note $\bigcup_j V^n_{i_j}$ cover $K_n \setminus K_{n-1} \Rightarrow \bigcup_n \bigcup_j V^n_{i_j}$ cover X.

$\forall x \in X$, $\exists n$ st $x \in \text{int}(K_n)$.

But $\text{int}(K_n)$ only meets $\bigcup_{m \le n} \bigcup_j V^m_{i_j}$ = finite.

$\Rightarrow V^n_{i_j}$ are locally finite refinement. □

Covering Dimension

Defn: X = space. The covering dimension of X is infimum over $n \in \mathbb{N}$ st $\forall\ \mathcal{U}_\alpha$ = open $\exists$ refinement V_β st $\forall x \in X$, x meets at most $n+1$ of the V_β.

Lemma: The covering dimension of $[0,1]^n$ is bdd by $2^n - 1$

Fact: The covering dim of $[0,1]^n$ is n

Lemma: (Lebesgue's covering lemma) X = cpt metric space, $\mathcal{U}_\alpha$ = open cover of X. $\exists\ \delta > 0$ st $\forall x$ there is α so that $x \in B_x(\delta) \subseteq \mathcal{U}_\alpha$.

Proof: Pass to a subcover $\mathcal{U}_1, \dots, \mathcal{U}_n$.

$B_\delta(x) \subseteq \mathcal{U}_i$ iff $d(x, X \setminus \mathcal{U}_i) \geq \delta$

Consider $f(x) = \max_i (d(x, X \setminus \mathcal{U}_i)) = $ cts

$X \setminus \mathcal{U}_i$ closed $\Rightarrow f(\mathcal{U}_i) > 0 \Rightarrow f > 0$ since $\mathcal{U}_i$ cover.

$X = $ cpt $\Rightarrow f(X) \subseteq (0, \infty)$ is cpt $\Rightarrow \exists$ lower bdd $f(X) \geq \delta$ □

Proof: Spse $\mathcal{U}_\alpha$ cover $[0,1]^n$. WLOG assume finite $\mathcal{U}_1, \dots, \mathcal{U}_N$

Divide $[0,1]$ as $\underbrace{[0, \tfrac{1}{m}]}_{I_1}, \underbrace{[\tfrac{1}{m}, \tfrac{2}{m}]}_{I_2}, \dots, \underbrace{[\tfrac{m-1}{m}, 1]}_{I_m}$ for $m >> 0$ so that

① $N_{i_1, \dots, i_m} = \{ x \mid d(x, I_{i_1} \times \dots \times I_{i_m}) < \delta \} \subseteq \mathcal{U}_K$.

② $x \in X$, then x only meets 2^n of the $N_{i_1, \dots, i_m}$

Take the cover $N_{i_1, \dots, i_m}$. □