Lecture # 1 – September 7th, 2023

Defn: A topology on a set X is a set of subsets of X, $\mathcal{O}$, called open sets st

1. $\phi, X \in \mathcal{O}$
2. $\mathcal{O}' \subseteq \mathcal{O} \Rightarrow \bigcup_{U \in \mathcal{O}'} U \in \mathcal{O}$
3. $U_1, \dots, U_n \in \mathcal{O} \Rightarrow \bigcap_{i=1}^{n} U_i \in \mathcal{O}$.

$(X, \mathcal{O})$ is a topological space

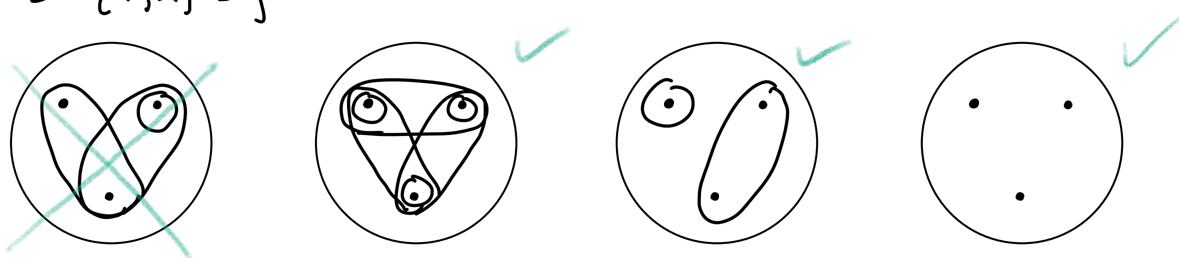
Remark:

- (ii) $\Rightarrow$ arb. unions of opens is open
- (iii) $\Rightarrow$ finite intersections of opens is open
- Often call X a (top.) space w/ $\mathcal{O}$ understood.

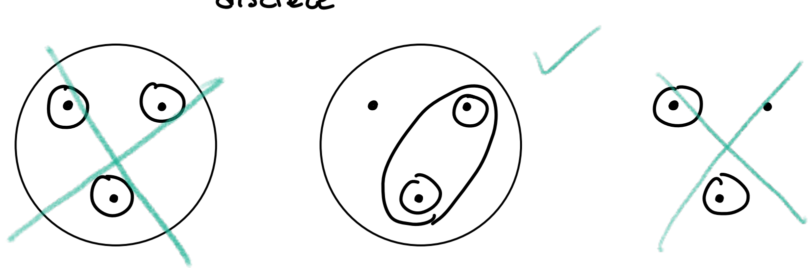
Defn: Spse $\mathcal{O}, \mathcal{O}'$ are two top. on X w/ $\mathcal{O} \subseteq \mathcal{O}'$.
$\mathcal{O}'$ is finer than $\mathcal{O}$ and $\mathcal{O}$ is coarser than $\mathcal{O}'$

Example:
1. The discrete top on X is $\mathcal{O}$ = power set of X
2. the trivial top on X is $\mathcal{O} = \{\phi, X\}$.

Example: $X = \{1, 2, 3\}$



discrete   triv.



Defn: $\mathcal{B}$ = set of subsets of X is a basis if

(i) $X = \bigcup_{B \in \mathcal{B}} B$

(ii) $x \in B' \cap B''$ w/ $B', B'' \in \mathcal{B} \Rightarrow \exists B \in \mathcal{B}$ st $x \in B \subseteq B_1 \cap B_2$

Lemma: A basis $\mathcal{B}$ generates a topology $\mathcal{O}$ via

$\mathcal{U} \in \mathcal{O}$ iff $\forall x \in \mathcal{U}, \exists B \in \mathcal{B}$ w/ $x \in B \subset \mathcal{U}$.

Proof: (i) Vacuously, ϕ is open

X is open since $\mathcal{B}$ covers X

(ii) Spse $\mathcal{U}_i \subseteq X$ open, $\exists B_i$ w/ $x \in B_i \subseteq \mathcal{U}_i$

If $x \in \bigcup_i \mathcal{U}_i \Rightarrow x \in \mathcal{U}_i$ for some i

$\Rightarrow x \in B_i \subseteq \bigcup_i \mathcal{U}_i$.

$\Rightarrow \bigcup_i \mathcal{U}_i$ open

(iii) $x \in \mathcal{U}_1 \cap \mathcal{U}_2 \Rightarrow \exists\ x \in B_1 \subset \mathcal{U}_1,\ x \in B_2 \subseteq \mathcal{U}_2$

$\Rightarrow \exists\ x \in B \subseteq B_1 \cap B_2 \subset \mathcal{U}_1 \cap \mathcal{U}_2$

$\Rightarrow \mathcal{U}_1 \cap \mathcal{U}_2$ open

By induction, $\bigcap_{i=1}^{n} \mathcal{U}_i$ open $\square$

Example: $X = \mathbb{R}$,

(1) The standard top on $\mathbb{R}$ has basis (a, b)

(2) Basis = $\{[a, b)\}$

(3) Basis = $\{\mathcal{U} \subset \mathbb{R} \mid \mathcal{U} = \mathbb{R} \setminus \{x_1, \ldots, x_n\}$ for some $n\}$

Note, (1) finer than (3)

(2) is finer than (1)

Rem: uncountable intersections need not be open.

$X = \mathbb{R}$ w/ std. top.

$\mathcal{U}_n = (-1/n, 1/n)$

$\bigcap_n \mathcal{U}_n = \{0\} \neq$ open.

Rem: Same top. could be generated by multiple different bases.

$X = \mathbb{R}^2$

① Basis $= \{B_x(r)\} = \{$open ball of radius r centered at $x\}$

② " $= \{Sq_x(r)\} = \{$open square w/ diag $2r$ " " $x\}$

Defn: X = space. $A \subset X$ is <u>closed</u> if $X \setminus A$ is open.

Ex: $X = \mathbb{R}$ w/ std top.: $[a,b]$, $[a,+\infty)$, $(-\infty,+\infty)$, $\{a\}$.

Lemma: (i) ϕ, X are closed

(ii) Any intersection of closed sets is closed

(iii) Finite union of closed sets is closed.

Proof: (i) ϕ = open $\Rightarrow X = X \setminus \phi$ = closed

X = open $\Rightarrow \phi = X \setminus X$ = closed

(ii) Write $A_i = X \setminus \mathcal{U}_i$ for $\mathcal{U}_i$ open w/ $i \in I$ = index set

$$\bigcap_{i \in I} A_i = \bigcap_{i \in I} (X \setminus \mathcal{U}_i) = \underbrace{X \setminus \bigcup_{i \in I} \mathcal{U}_i}_{\text{de Morgan's laws}} = \text{closed}$$

(iii)

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} (X \setminus \mathcal{U}_i) \underset{\text{de Morgan's Laws}}{=} X \setminus \bigcap_{i=1}^{n} \mathcal{U}_i = \text{closed}$$

□

Defn: X = top., $A \subseteq X$.

(i) $\mathrm{int}(A) = \bigcup_{\substack{\mathcal{U} \in \mathcal{O} \\ \mathcal{U} \subseteq A}} \mathcal{U}$ = interior of A

(ii) $\bar{A} = \bigcap_{\substack{C \text{ closed} \\ A \subseteq C}} C$ = closure of A

Ex: X = discrete top., $A \subset X$ is open + closed $\Rightarrow \bar{A} = A = \mathrm{int}(A)$

X = triv. top, $\emptyset \neq A \subsetneq X \Rightarrow \bar{A} = X$, $\mathrm{int}(A) = \emptyset$

Lemma: (i) A is open iff $A = \mathrm{int}(A)$

" " closed " $A = \bar{A}$

Proof: (i) $\mathrm{int}(A) = \bigcup_{\mathcal{U} \subseteq A} \mathcal{U} = A \cup \bigcup_{\mathcal{U} \subsetneq A} \mathcal{U} = A$

(ii) $\bar{A} = \bigcap_{A \subseteq C} C = A \cap \bigcap_{A \subsetneq C} C = A$ $\square$

Defn: $A \subseteq X$ = top is <u>dense</u> if $\bar{A} = X$.

Ex: $\mathbb{R} \setminus \{1, 2, 3, \dots\}$ is dense in $\mathbb{R}$

$\mathbb{Q}$ is dense in $\mathbb{R}$.

Warning: A, B = dense $\not\Rightarrow$ $A \cap B$ dense

$\hookrightarrow$ $\mathbb{Q} \subseteq \mathbb{R}$, $\mathbb{Q} + \sqrt{2} \subseteq \mathbb{R}$.

Defn: (i) $x \in X$ is a limit point of A if $\forall\, x \in \mathcal{U} \in \mathcal{O}$, $A \cap \mathcal{U} \neq \emptyset$.

(iii) $\partial A = \{ x \in X \mid x \text{ limit pt of } A \;\&\; X \setminus A \}$.

Lemma: ① $\bar{A} = \{\text{limit points of } A\} = \text{int}(A) \cup \partial A$

② $X = \text{int}(A) \sqcup \partial A \cup \text{int}(X \setminus A)$ (Exer)

Proof:

- $\bar{A} \subseteq \{\text{limit pts}\}$

  $x \in \bar{A}$ and spse $x \notin \{\text{lim}\}$

  $\Rightarrow \exists\, U \ni x$ w/ $U \cap A = \emptyset$

  $\Rightarrow A \subseteq X \setminus U = \text{closed}$

  $\Rightarrow \bar{A} \subseteq X \setminus U \quad \Rightarrow\Leftarrow$

- $\{\text{lim}\} \subseteq \text{int}(A) \sqcup \partial A$

  $x \in \text{lim} \Rightarrow \forall\, U \ni x,\ U \cap A \neq \emptyset$

  if $\exists\ x \in U \subset A \Rightarrow x \in \text{int}(A)$

  else $\forall\, U \ni x,\ U \cap (X \setminus A) \neq \emptyset \Rightarrow x \in \partial A$.

- $\text{int}(A) \sqcup \partial A \subseteq \bar{A}$

  $x \in \partial A$, spse $x \notin \bar{A}$

  $\Rightarrow \exists\ x \in X \setminus \bar{A} = \text{open}$

  $\Rightarrow (X \setminus \bar{A}) \cap A \subset (X \setminus A) \cap A = \emptyset \quad \Rightarrow\Leftarrow$ $\square$

Lemma: If U_0, U_1 = dense opens $\Rightarrow U_0 \cap U_1$ = dense open.