1. Semiclassical calculus

We first briefly discuss semiclassical pseudodifferential operators on $\mathbb{R}^n$ and on compact manifolds.

1.1. On Euclidean space. We consider symbols depending on the semiclassical parameter $h \in (0, 1)$ smoothly down to $h = 0$, $a = a(h, x, \xi) \in C^\infty([0, 1)_h; S^m(\mathbb{R}_x^n; \mathbb{R}_\xi^n))$. (1.1)

The semiclassical quantization of $a$ is an $h$-dependent family of operators on $\mathcal{S}'(\mathbb{R}^n)$,

$$\text{Op}_h(a) := \frac{1}{(2\pi h)^n} \int \int e^{i(x-y) \cdot \xi/h} a(h, x, \xi) u(y) \, dy \, d\xi$$

$$= \frac{1}{(2\pi)^n} \int \int e^{i(x-y) \cdot \xi} a(h, x, h\xi) u(y) \, dy \, d\xi.$$ (1.2)

For any fixed $h > 0$, this is thus an element of $\Psi^m(\mathbb{R}^n)$, but there is extra structure in the limit $h \downarrow 0$, as we shall discuss momentarily. We define the space of $m$-th order semiclassical ps.d.o.s by

$$\Psi^m_h(\mathbb{R}^n) := \{ (\text{Op}_h(a))_{h \in (0, 1)} ; a \in C^\infty([0, 1)_h; S^m(\mathbb{R}_x^n; \mathbb{R}_\xi^n)) \}. \quad (1.3)$$

One often saves space and writes $A = (\text{Op}_h(a))_{h \in (0, 1)}$; operations such as taking adjoints or compositions are then defined individually for each $h$. That is, $A^* = (\text{Op}_h(a)^*)$, and if $B = (\text{Op}_h(b))_{h \in (0, 1)}$, then $A \circ B = (\text{Op}_h(a) \circ \text{Op}_h(b))_{h \in (0, 1)}$.

Paralleling the development of the standard calculus of ps.d.o.s on $\mathbb{R}^n$, one can show that the class of semiclassical ps.d.o.s is closed under composition and taking adjoints. The key ingredient is again the left reduction formula for a symbol $a = a(h, x, y, \xi) \in C^\infty([0, 1)_h; S^m(\mathbb{R}_x^n \times \mathbb{R}_y^n; \mathbb{R}_\xi^n))$. Directly from the non-semiclassical discussion, one has $\text{Op}_h(a) = \text{Op}_h(a_L)$ with

$$a_L(h, x, \xi) \sim \sum_{\alpha \in \mathbb{N}^n_0} \frac{1}{\alpha!} (\partial^\alpha_{\xi}(hD_y)^{\alpha} a(h, x, y, \xi))|_{y=x}$$ (1.4)

for each fixed $h$. Note now that the $\alpha$-th term in the expansion comes with a factor $h^{|\alpha|}$, i.e. it lies in $h^{|\alpha|}C^\infty([0, 1)_h; S^{m-|\alpha|})$. One can then show that the left reduction statement (1.4)
can be strengthened; namely one has \( \text{Op}_h(a) = \text{Op}_h(a_L) \) where \( a_L \in C^\infty([0, 1); S^m) \) is an asymptotic sum (1.4) in the sense that for all \( N \)

\[
a_L(h, x, \xi) = \sum_{\alpha \in \mathbb{N}_0^n, |\alpha| < N} \frac{1}{\alpha!} (\partial^\alpha_x (hD_y)^\alpha a(h, x, y, \xi)) \big|_{y = x} \in h^N C^\infty([0, 1); S^{m-N}). \tag{1.5}
\]

Thus, \( a_L \) is unique modulo the space

\[
h^\infty C^\infty([0, 1); S^{-\infty}) = \bigcap_{N \in \mathbb{N}} h^N C^\infty([0, 1); S^{-N})
\]

of residual symbols which vanish rapidly as \( |\xi| \to \infty \), and which also vanish to infinite order at \( h = 0 \).

Similarly then, the full left symbol \( c \) of \( A \circ B \) where \( A = (\text{Op}_h(a)), B = (\text{Op}_h(b)) \), is given by an asymptotic sum

\[
c(h, x, \xi) \sim \sum_{\alpha \in \mathbb{N}_0^m} \frac{1}{\alpha!} \partial^\alpha_x a(h, x, \xi) (hD_x)^\alpha b(h, x, \xi);
\]

notice again the gain of powers of \( h \).

In conclusion, the principal symbol map is now

\[
\sigma^m_h : \Psi^m_h(\mathbb{R}^n) \to C^\infty([0, 1)_h; S^m(\mathbb{R}^n; \mathbb{R}^n))/hC^\infty([0, 1)_h; S^{m-1}(\mathbb{R}^n; \mathbb{R}^n)). \tag{1.8}
\]

Note that the restriction to \( h = 0 \) is an honest symbol in \( S^m \), not merely an equivalence class of symbols! From now on, we shall pass to standard notation and drop the ‘\( C^\infty([0, 1)_h; \cdots] \)’ bit, thus writing \( S^m/hS^{m-1} \) for the right hand side of (1.8).

For \( A \in \Psi^m_h(\mathbb{R}^n), B \in \Psi^{m'}_h(\mathbb{R}^n) \), we have \( \sigma^{m+m'}_h(A \circ B) = \sigma^m_h(A)\sigma^{m'}_h(B) \) as usual; but notice carefully that this entails the multiplicativity of the full symbols at \( h = 0 \) as well.

The commutator formula now becomes

\[
\frac{i}{h} [A, B] \in \Psi^{m+m'-1}_h(\mathbb{R}^n), \quad \sigma^{m+m'-1}_h\left(\frac{i}{h} [A, B]\right) = \{\sigma^m_h(A), \sigma^{m'}_h(B)\}. \tag{1.9}
\]

Examples of semiclassical differential operators (quantizations of symbols which are polynomials in \( \xi \)) on \( \mathbb{R}^n \) are

\[
1 \in \Psi^0_h, \quad hD_{x_1} \in \Psi^1_h, \quad h^2 \Delta + V \in \Psi^2_h, \tag{1.10}
\]

where \( V \in C^\infty_c(\mathbb{R}^n) \). The principal symbols of these operators are 1, \( \xi_1, |\xi|^2 + V(x) \). Notice carefully that \( V(x) \), despite not involving any differentiation, is seen on the principal symbol level at \( h = 0 \)! (See that you understand why, for example by computing the composition \( hD_{x_1} \circ (h^2 \Delta + V) \) and its principal symbol as an element of \( \Psi^2_h \).

The associated Sobolev spaces are denoted

\[
H^s_h(\mathbb{R}^n) := \{ u \in \mathcal{S}'(\mathbb{R}^n) : (hD)^s u \in L^2(\mathbb{R}^n) \}, \tag{1.11}
\]

with norm \( \|u\|_{H^s_h}^2 := \|(hD)^s u\|_{L^2}^2 \). Thus, \( H^s_h(\mathbb{R}^n) = H^s(\mathbb{R}^n) \) as sets, but the norms are not equivalent uniformly as \( h \to 0 \). We also define weighted versions of these spaces,

\[
h^p H^s_h(\mathbb{R}^n), \quad \|u\|_{h^p H^s_h(\mathbb{R}^n)} := \|h^{-p} u\|_{H^s_h(\mathbb{R}^n)}. \tag{1.12}
\]
1.2. On compact manifolds. Mirroring our development of standard ps.d.o.s, one can transfer the semiclassical calculus on $\mathbb{R}^n$ to (possibly noncompact) smooth manifolds. We shall restrict to the case of $n$-dimensional closed manifolds $M$ (compact, no boundary) here. Thus, one can define the space

$$\Psi^m_h(M)$$

of $h$-dependent ($h \in (0, 1)$) bounded linear operators $A = (A_h)_{h \in (0,1)}$ on $C^\infty(M)$ which in a local coordinate chart near the diagonal of $M \times M$ are given by an element of $\Psi^m_h(\mathbb{R}^n)$, whereas for $\phi, \psi \in C^\infty(M)$ with disjoint supports one requires $\phi A \psi = (\phi A_h \psi)_{h \in (0,1)}$ to have Schwartz kernel in $h^\infty C^\infty([0,1] \times M \times M; \Omega_R)$ where $\Omega_R$ is the right 1-density bundle. Thus, off the diagonal, the Schwartz kernel of $A$ is not merely smooth, but in addition vanishes to infinite order as $h \to 0$. The principal symbol map is

$$\sigma^m_h: \Psi^m_h(M) \to C^\infty([0,1)_h; S^m(T^*M))/hC^\infty((0,1)_h; S^{m-1}(T^*M)),$$

which, again, is usually written in the more condensed form $\sigma^m_h: \Psi^m_h(M) \to S^m/hS^{m-1}$. The space of residual operators is $h^\infty \Psi^{-\infty}_h(M)$. Schwartz kernels of elements of this class are smooth functions on $[0,1)_h \times M^2$, which vanish to infinity order at $h = 0$.

1.2.1. Traces. Consider an element $A \in \Psi^{-\infty}_h(M)$. Carefully note that its principal symbol (restricted to $h = 0$) is a well-defined element $a \in S^{-\infty}(T^*M) = \mathscr{S}(T^*M)$. On the other hand, the Schwartz kernel $K_{A_h} \in \mathcal{C}(\mathbb{R} \times M; \Omega_R)$ of $A_h$ is smooth for each fixed $h > 0$. Fixing a volume density $d\mu$ on $M$ and any orthonormal basis $e_1, e_2, \ldots \in L^2(M)$ of $L^2(M)$, we define the trace of $A_h$ as

$$\text{tr}(A_h) := \sum_j \langle A_h e_j, e_j \rangle_{L^2(M)} = \sum_j \int_M K_{A_h}(x,y)e_j(y)\overline{e_j(x)} \, d\mu(x). \quad (1.13)$$

(From the expression on the right one easily sees that the series converges.) Since the operators with Schwartz kernel $K_N(x,y) := \sum_{j=1}^N e_j(x)e_j(y) \, d\mu(x) \in L^2(M \times M; \Omega_L)$ converge strongly to the identity map on $L^2(M \times M; \Omega_M)$ as $N \to \infty$, we have distributional convergence $K_N \to \delta_{\Delta}$ where $\Delta \subset M \times M$ is the diagonal, and $\delta_{\Delta}$ is the (left 1-density) $\delta$-distribution on $\Delta$. Therefore, we obtain from (1.13) the equivalent description

$$\text{tr}(A_h) = \int_M K_{A_h}(x,x). \quad (1.14)$$

(Note here that the restriction of $K_{A_h}$ to $\Delta$ is a 1-density on $\Delta \cong M$, hence can be invariantly integrated indeed.) In particular, $\text{tr}(A_h)$ is independent of the choice of volume density and orthonormal basis. Equating (1.14) with (1.13) for problem-specific $e_j$ can give a great deal of information, as we will see in the proof of Weyl’s law below.$^1$

Lastly, we relate $\text{tr}(A_h)$ to the semiclassical principal symbol $a$.

Lemma 1.1. For $A = (A_h)_{h \in (0,1)} \in \Psi^{-\infty}_h(M)$ and $a = \sigma_h(A) \in S^{-\infty}(T^*M)$, we have

$$\text{tr}(A_h) = \frac{1}{(2\pi \hbar)^n} \int_{T^*M} a(x,\xi) \, dx \, d\xi + \mathcal{O}(\hbar^{-n+1}), \quad h \to 0. \quad (1.15)$$

Here, $dx \, d\xi$ is the canonical volume density on $T^*M$ (induced by the canonical symplectic form).

$^1$For a proper functional analytic treatment of the trace, and the associated class of trace class operators, we refer the reader to the literature, e.g. [Tay11, Appendix A.6].
Proof. Upon evaluating the trace via (1.14), it suffices to prove this in local coordinates. Thus, suppose \( a \in C^\infty((0,1)_h; S^{-\infty}(\mathbb{R}^n_x; \mathbb{R}^n_\xi)) \) has compact support in \( x \). The Schwartz kernel \( K_h \) of \( \text{Op}_h(a) \) is

\[
K_h(x,y) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot \xi/h} a(h,x,\xi) \, d\xi,
\]

the restriction of which to the diagonal \( y = x \) is

\[
K_h(x,x) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} a(h,x,\xi) \, d\xi.
\]

Integration in \( x \) gives

\[
\text{tr}(\text{Op}_h(a)) = (2\pi h)^{-n} \left( \int \int a(0,x,\xi) \, dx \, d\xi + \mathcal{O}(h) \right).
\]

But \( a(0,x,\xi) \) is the semiclassical principal symbol of \( (\text{Op}_h(a)) \), hence the Lemma follows. \( \square \)

1.2.2. Functional calculus. We denote by \( M \) a closed manifold, and we fix a volume density on \( M \). Let \( m > 0 \), and suppose \( P = (P_h)_{h \in (0,1)} \in \Psi_h^m(M) \) is a symmetric operator, with semiclassical principal symbol \( p \in S^m(T^*M) \) so that \( [p] \in S^m/S^{m-1} \) is elliptic. Denote by \( e_j(h) \in L^2(M) \) a complete orthonormal basis of eigenfunctions of \( P_h \), with corresponding eigenvalues \( E_j(h) \in \mathbb{R} \).

For \( f \in C_c^\infty(\mathbb{R}) \), we then define an operator \( f(P) = (f(P_h))_{h \in (0,1)} \) on \( L^2(M) \) by

\[
f(P_h)e_j(h) := f(E_j(h))e_j(h).
\]

The following result shows that this abstractly (spectral theoretically) defined operator in fact is a semiclassical ps.d.o., and computes its principal symbol:

**Proposition 1.2.** We have \( f(P) \in \Psi^{-\infty}_h(M) \), and \( \sigma_h(f(P)) = f(\sigma_h(P)) = f(p) \).

The starting point of the proof is the following rewriting of (1.19).

**Lemma 1.3** (Almost analytic extensions). Let \( f \in C_c^\infty(\mathbb{R}) \). Then there exists an almost analytic extension \( \tilde{f} \in C_c^\infty(\mathbb{C}) \) of \( f \), which means that

1. \( \text{supp} \, \tilde{f} \subset \{ z \in \mathbb{C} : |\text{Im} \, z| \leq 1 \} \),
2. \( \tilde{f}|_{\mathbb{R}} = f \),
3. \( \partial_z \tilde{f}(z) = \mathcal{O}(|\text{Im} \, z|^{\infty}) \) where \( \partial_z = \frac{1}{2}(\partial_x + i\partial_y) \) (writing \( z = x + iy \)), i.e. the anti-holomorphic derivative of \( \tilde{f} \) vanishes to infinite order at \( \mathbb{R} \).

**Proof.** Pick \( \chi \in C_c^\infty((-1,1)) \) to be identically 1 on \( [-\frac{1}{2}, \frac{1}{2}] \). For any \( N \), the function

\[
\tilde{f}_N(x + iy) := \sum_{j=0}^N \frac{f^{(j)}(x)}{j!} (iy)^j \chi(y)
\]

satisfies all requirements except that only \( \partial_z \tilde{f}_N(z) = \mathcal{O}(|\text{Im} \, z|^N) \). In order to construct the desired \( \tilde{f} \), one uses a Borel type construction,

\[
\tilde{f}(x + iy) := \sum_{j=0}^\infty \frac{f^{(j)}(x)}{j!} (iy)^j \chi(e_j^{-1} y),
\]
where $\epsilon_j \searrow 0$ sufficiently rapidly as $j \to \infty$. 

**Lemma 1.4** (Helffer–Sjöstrand formula). Let $f \in C_c^\infty(\mathbb{R})$, and denote by $\tilde{f}$ an almost analytic extension of $f$. Then

$$f(P_h) = \frac{1}{\pi i} \int_{\mathbb{C}} \partial_{\tilde{z}} \tilde{f}(z)(P_h - z)^{-1} \, d\lambda,$$

(1.22)

where $d\lambda$ is the Lebesgue measure on $\mathbb{C}$.

**Proof.** It suffices to verify this formula for $P_h$ replaced by an eigenvalue $E_j(h)$, or indeed by any real number $t$. In that case, it follows from the fact that $\partial_{\tilde{z}}(t - z)^{-1} = 0$ for $z \neq t$, and

$$\frac{1}{\pi i} \int_{\mathbb{C}} \partial_{\tilde{z}} \tilde{f}(z)(t - z)^{-1} \, d\lambda = \frac{1}{\pi i} \lim_{\epsilon \to 0} \int_{\mathbb{C} \setminus B(t,\epsilon)} \partial_{\tilde{z}}(\tilde{f}(z)(t - z)^{-1}) \, d\lambda$$

$$= \frac{1}{2\pi i} \lim_{\epsilon \to 0} \oint_{\partial B(t,\epsilon)} \tilde{f}(z)(t - z)^{-1} \, dz$$

$$= \frac{1}{2\pi i} \lim_{\epsilon \to 0} \oint_{\partial B(t,\epsilon)} (f(t) + O(\epsilon))(t - z)^{-1} \, dz$$

$$= f(t). \quad \Box$$

**Proof of Proposition 1.2.** Note that for $z \notin \mathbb{R}$, the operator $P - z \in \Psi_h^m(M)$ has elliptic semiclassical principal symbol $p - z$, and indeed $|p - z| \geq |\text{Im } z|$, and $|p(x, \xi) - z| \geq c|\xi|^m$ for some $c > 0$ for large $\xi$. For any $k \in \mathbb{N}$, we can thus perform the elliptic parametrix construction for $P - z$ to order $k$, giving

$$B_k(z) \in \Psi_h^{-m}(M), \quad (P - z) \circ B_k(z) = I, \quad B_k(z) \circ (P - z) = I \in h^k\Psi_h^{-k}(M),$$

(1.23)

and with the sum of the first $k$ seminorms of these three operators bounded by $C_k|\text{Im } z|^{-c_k}$ for some constants $c_k, C_k$.

We define what we hope to be an approximation of $f(P)$ by setting

$$F_k := \frac{1}{\pi i} \int_{\mathbb{C}} \partial_{\tilde{z}} \tilde{f}(z)B_k(z) \, d\lambda \in \Psi_h^{-m}(M).$$

(1.24)

Note that $F_{k+1} - F_k$ is an integral of the same type, but with $B_{k+1} - B_k \in h^k\Psi_h^{-m-k}(M)$ in the integrand; hence

$$F_{k+1} - F_k \in h^k\Psi_h^{-m-k}(M).$$

(1.25)

Consider on the other hand the difference

$$F_k - f(P) = \frac{1}{\pi i} \int_{\mathbb{C}} \partial_{\tilde{z}} \tilde{f}(z)(B_k(z) - (P - z)^{-1}) \, d\lambda.$$

In view of (1.23), and since $\|(P - z)^{-1}\|_{L^2 \to L^2} \leq |\text{Im } z|^{-1}$, we conclude that

$$F_k - f(P) \in \mathcal{L}(L^2(M), h^kH_h^k(M)) \cap \mathcal{L}(h^{-k}H_h^{-k}(M), L^2(M))$$

$$\subset \mathcal{L}(h^{-k/2}H_h^{-k/2}(M), h^{k/2}H_h^{k/2}(M))$$

(1.27)

is a regularizing operator, gaining $k$ semiclassical derivatives and $k$ powers of $h$. 


Thus, let us take $\tilde{F} \in \Psi^{-m}_h(M)$ to be an asymptotic sum

$$\tilde{F} \sim F_0 + \sum_{k=0}^{\infty} (F_{k+1} - F_k).$$

Then $\tilde{F} - f(P)$ is a completely regularizing operator, mapping $h^{-\infty}H^\infty_h(M) \to h^\infty H^\infty_h(M)$. We leave it to the reader to verify that this implies that the Schwartz kernel of $\tilde{F} - f(P)$ lies in $h^\infty C^\infty((0,1)_h \times M^2)$, i.e. $\tilde{F} - f(P) \in h^\infty \Psi^{-\infty}_h(M)$. So far, we have shown that

$$f(P) = \tilde{F} - (\tilde{F} - f(P)) \in \Psi^{-m}_h(M).$$

But for any $N \in \mathbb{N}$, we can write $f(x) = (x + i)^{-N} f_N(x)$ with $f_N \in C^\infty_c(\mathbb{R})$; then

$$f(P) = (P + i)^{-N} f_N(P) \in \Psi^{-Nm}_h(M) \circ \Psi^{-m}_h(M) \subset \Psi^{-(N+1)m}_h(M).$$

(Here, we used that $P + i$, being an elliptic semiclassical ps.d.o., is invertible for all sufficiently small $h > 0$, with inverse an element of $\Psi^{-m}_h(M)$. Indeed, the elliptic parametrix construction gives $Q \in \Psi^{-m}_h(M)$ so that $(P + i)Q = I + R$ with $R$ having Schwartz kernel in $h^\infty C^\infty((0,1) \times M^2)$. Hence, $R$ is small as an operator on $L^2(M)$, and thus $I + R$ is invertible on $L^2(M)$ via Neumann series, with $(I + R)^{-1} = I + R$ for some $R \in h^\infty C^\infty$. Thus $(P + i)^{-1} = Q(I + R) \in \Psi^{-m}_h(M)$.) Therefore, $f(P) \in \Psi^{-\infty}_h(M)$.

The principal symbol is computed directly using Lemma 1.4 by pushing the principal symbol map $\sigma_h$ through the integral sign and using that $\sigma^{-m}_h((P - z)^{-1}) = (p - z)^{-1}$, as follows directly from the parametrix construction.  

$$\square$$

2. Weyl’s law

We denote by $M$ a closed manifold. Fix a volume density on $M$.

**Theorem 2.1.** Let $m > 0$, and suppose $P = (P_h)_{h \in (0,1)} \in \Psi^m_h(M)$ is a symmetric operator, with semiclassical principal symbol $p \in S^m(T^*M)$ so that $[p] \in S^m/S^{m-1}$ is elliptic. Denote the eigenvalues of $P_h$ by $E_1(h), E_2(h), \ldots$. Then for any $a < b \in \mathbb{R}$ for which $(a', b') \mapsto \text{vol}_{T^*M}(p^{-1}([a', b']))$ is continuous at $(a', b') = (a, b)$, we have

$$\# \{ j : E_j(h) \in [a, b] \} = \frac{1}{(2\pi h)^n} \left( \text{vol}_{T^*M}(\{ a \leq p(x, \xi) \leq b \} \big) + o(1) \right). \hspace{1cm} (2.1)$$

A typical example is $P = h^2 \Delta_g + V$ where $V \in C^\infty(M)$ is a real-valued potential. In the special case $V = 0$, we obtain:

**Corollary 2.2.** Let $g$ denote a smooth Riemannian metric on $M$, and let $d = \text{dim } M$. Denote by $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \to \infty$ the eigenvalues of the Laplace–Beltrami operator $\Delta_g$ on $M$. Then

$$\# \{ j : \lambda_j \leq N \} = \frac{\omega_d \text{vol}(M)}{(2\pi)^d} N^{d/2} + o(N^{d/2}), \hspace{1cm} (2.2)$$

where $\omega_d = \pi^{d/2}/\Gamma(\frac{d}{2} + 1)$ denotes the volume of the unit ball in $\mathbb{R}^n$. 


Proof. Apply the Theorem to \( P_h = h^2 \Delta_g \in \Psi^2_1(M) \), \( p(x, \xi) = |\xi|^{2-1(x)} \), and note that the eigenvalues of \( P_h \) are \( E_j(h) = h^2 \lambda_j \), \( j \in \mathbb{N}_0 \). Thus, taking \( a = 0 \) and \( b = 1 \), and \( h = N^{-1/2} \), the left hand sides of (2.1) and (2.2) coincide, and the right hand side of (2.2) comes from (2.1) by integrating out in the fiber (\( \xi \)) variable. □

Proof of Theorem 2.1. Let \( f_1, f_2 \in C^\infty_0(\mathbb{R}) \) be such that \( f_1(x) \leq 1_{[a,b]}(x) \leq f_2(x) \), where \( 1_{[a,b]}(x) \) is the characteristic function of the interval \([a, b]\). Then, by evaluating the trace using the formula (1.13) where the \( e_j \) is an \( L^2 \)-normalized eigenfunctions of \( P_h \) corresponding to the eigenvalue \( E_j(h) \), we find

\[
tr f_1(P_h) \leq \#\{ j : E_j(h) \in [a,b] \} \leq tr f_2(P_h). \tag{2.3}
\]

But since \( f_k(P) \in \Psi^{-\infty}_h(M) \), \( k = 1, 2 \), has semiclassical principal symbol \( f_k(p) \) by Proposition 1.2, we also have

\[
tr f_k(P_h) = \frac{1}{(2\pi h)^n} \left( \int_{T^*M} f_k(p(x, \xi)) \, dx \, d\xi \right) + O(h^{-n+1}). \tag{2.4}
\]

The Theorem follows from these observations as follows: given \( \epsilon > 0 \), we can pick \( f_1, f_2 \) as above to be regularizations of the characteristic function \( 1_{[a,b]} \) in such a way that

\[
\left| \int_{T^*M} f_k(p(x, \xi)) \, dx \, d\xi - \text{vol}_{T^*M}(p^{-1}([a,b])) \right| < \epsilon. \tag{2.5}
\]

This then implies by (2.3) and (2.4) that

\[
\left| (2\pi h)^n \#\{ j : E_j(h) \in [a,b] \} - \text{vol}_{T^*M}(p^{-1}([a,b])) \right| \leq \epsilon + C_\epsilon h < 2\epsilon \tag{2.6}
\]

(with \( C_\epsilon h \) bounding \( (2\pi h)^n \) times the \( O(h^{-n+1}) \) error term in (2.4)) for all sufficiently small \( h > 0 \). □

References


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