18.157 (S21): HOMEWORK 3

Due: Sunday, Apr 11, by midnight.¹ Send your pdf (LaTeX preferred, or handwritten but legible!) to phintz@mit.edu.

We continue the study of scattering pseudodifferential operators.

**Exercise 1.** (Scattering ps.d.o.s, VII: boundedness on Sobolev spaces.)

1. Prove that elements of $\Psi^0_{sc}(\mathbb{R}^n)$ are bounded as maps on $L^2(\mathbb{R}^n)$.
2. Show that $\Lambda_{m,r} := \langle x \rangle^r \langle D \rangle^m \in \Psi^{m,r}_{sc}(\mathbb{R}^n)$ and $\Lambda'_{m,r} := \langle D \rangle^m \langle x \rangle^r \in \Psi^{m,r}_{sc}(\mathbb{R}^n)$.
3. Let $A \in \Psi^{m,r}_{sc}(\mathbb{R}^n)$. Show that for all $\rho, \sigma \in \mathbb{R}$, $A$ is a bounded operator $A: \langle x \rangle^\rho H^\sigma(\mathbb{R}^n) \to \langle x \rangle^\rho + r H^{\sigma-m}(\mathbb{R}^n)$. (0.1)

**Exercise 2.** (Scattering ps.d.o.s, VIII: elliptic scattering ps.d.o.s are Fredholm.)

1. Let $m < m'$ and $r > r'$. Show that the inclusion $\langle x \rangle^{r'} H^{m'}(\mathbb{R}^n) \hookrightarrow \langle x \rangle^{r} H^{m}(\mathbb{R}^n)$ is compact.
2. Let $A \in \Psi^{m,r}_{sc}(\mathbb{R}^n)$ be elliptic—recall that this means that its principal symbol as a scattering ps.d.o. $\sigma^{m,r}_{sc}(A) \in S^{m,r}/S^{m-1,r-1}(T^*\mathbb{R}^n)$ is elliptic. Show that for any $\rho, \sigma \in \mathbb{R}$, the operator $A: \langle x \rangle^\rho H^\sigma(\mathbb{R}^n) \to \langle x \rangle^\rho + r H^{\sigma-m}(\mathbb{R}^n)$ (0.2)

 has finite-dimensional nullspace $\ker A$, and that $\ker A$ is independent of $\rho, \sigma$, and indeed $\ker A \subset \mathcal{S}(\mathbb{R}^n)$.
3. **Bonus exercise.** Show that the map (0.2) is Fredholm. Prove that its index $\text{ind} A = \dim \ker A - \dim \text{coker} A$ is independent of $\rho, \sigma$.

**Exercise 3.** Let $\Gamma \subset \mathbb{C}$ be a smooth, simple, closed curve. That is, $\Gamma = \gamma(S^1)$ where $\gamma: S^1 \to \mathbb{C}$ is smooth, injective, and regular ($\gamma'(t) \neq 0$ for all $t$). Let $K \in C^\infty(\Gamma \times \Gamma)$. Prove that

$$ Au(t) := \lim_{\epsilon \to 0} \int_{|t-s| \geq \epsilon} \frac{K(t, s)}{t-s} u(s) \, ds, \quad u \in C^\infty(\Gamma) $$ (0.3)

is well-defined and defines an element $A \in \Psi^0_{\Omega_1}(\Gamma)$. Compute its principal symbol.

**Exercise 4.** Let $M$ be a compact manifold, let $E, F \to M$ denote two vector bundles, and let $A \in \Psi^m(M; E, F)$.

1. Suppose $\sigma_m(A)$ is injective; that is, there exists $b \in S^{-m}(T^*M; \text{Hom}(F, E))$ such that $b\sigma_m(A) - 1 \in S^{-1}(T^*M; \pi^* \text{End}(E))$. Show that $A: H^s(M; E) \to H^{s-m}(M; F)$ has finite-dimensional kernel and closed range.

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¹See the course website, https://math.mit.edu/~phintz/18.157-S21/, for homework policies.
²The integrand involves multiplication with and division by complex numbers.
Exercise 5. (Helmholtz decomposition.) Let \((M, g)\) be a compact Riemannian manifold, denote by \(d: \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M; T^*M)\) the exterior derivative acting on functions, and denote by \(\delta_g = d^*\) its adjoint. Let \(\omega \in H^s(M; T^*M)\) be a 1-form. Prove that there exist \(u \in H^{s+1}(M)\) and \(\eta \in H^s(M; T^*M)\) such that
\[
\omega = du + \eta, \quad \delta_g \eta = 0. \tag{0.4}
\]

Exercise 6. Let \(M\) be a compact manifold, let \(E_i \to M, i = 0, \ldots, N\), be complex vector bundles, and suppose \(d_i \in \text{Diff}^1(M; E_i, E_{i+1}), i = 0, \ldots, N - 1\). Suppose they form a complex of differential operators
\[
\mathcal{C}^\infty(M; E_0) \xrightarrow{d_0} \mathcal{C}^\infty(M; E_1) \xrightarrow{d_1} \cdots \xrightarrow{d_{N-1}} \mathcal{C}^\infty(M; E_N); \tag{0.5}
\]
that is, for each \(i < N\),
\[
d_{i+1} \circ d_i = 0 \in \text{Diff}^2(M; E_i, E_{i+2}). \tag{0.6}
\]

Assume moreover that this complex is elliptic, meaning that the symbol complex
\[
\mathcal{C}^\infty(T^*M \setminus \partial; \pi^*E_0) \xrightarrow{\sigma_1(d_0)} \mathcal{C}^\infty(T^*M \setminus \partial; \pi^*E_1) \xrightarrow{\sigma_1(d_1)} \cdots \xrightarrow{\sigma_1(d_{N-1})} \mathcal{C}^\infty(T^*M \setminus \partial; \pi^*E_N) \tag{0.7}
\]
is exact (that is, \(\text{ran} \sigma_1(d_{i-1})(x, \xi) = \ker \sigma_1(d_i)\) for all \(i < N\)). The goal of this exercise is to study the cohomology groups
\[
H^i(E_\bullet) := (\ker d_i)/\text{ran} d_{i-1}, \quad i = 1, \ldots, N - 1, \tag{0.8}
\]
using PDE theory.

1. Equip \(M\) with a volume density and the \(E_i\) with Hermitian fiber inner products; define \(\delta_i \in \text{Diff}^1(M; E_i, E_{i-1})\) to be the adjoint of \(d_{i-1}\). Show that the ‘Laplacian’
\[
\Delta_i := d_{i-1} \circ \delta_i + \delta_{i+1} \circ d_i \in \text{Diff}^2(M; E_i), \quad 1 \leq i \leq N - 1, \tag{0.9}
\]
is elliptic and symmetric.

2. Show that
\[
\ker \Delta_i = \{u \in \mathcal{C}^\infty(M; E_i) : d_i u = 0, \ \delta_i u = 0\}. \tag{0.10}
\]

3. Show that the inclusion \(\ker \Delta_i \hookrightarrow \ker d_i\) induces an isomorphism of vector spaces
\[
\ker \Delta_i \cong H^i(E_\bullet). \tag{0.11}
\]

4. Prove the Hodge theorem: if \((M, g)\) is a compact Riemannian manifold, and \(\Delta_k \in \text{Diff}^2(M; \Lambda^k T^*M)\) is the Hodge Laplacian on \(k\)-forms, then \(\ker \Delta_k \cong H^k(M)\), where \(H^k(M)\) denotes the \(k\)-th de Rham cohomology group (with complex coefficients) of \(M\).