For simplicity, we will only consider scalar equations, and smooth domains in \( \mathbb{R}^n \). Extending the discussion to cover boundary value problems posed on smooth manifolds with boundary and for sections of vector bundles requires largely notational (and minor technical) modifications. We follow the presentation of [Hör07, §20] very closely.

1. Setup

Fix a compact domain
\[ X \subset \mathbb{R}^n \] (1.1)
with smooth boundary. We denote its interior by \( X^\circ \). We wish to study boundary value problems for an elliptic differential operator
\[ P \in \text{Diff}^m(X). \] (1.2)
Given \( u \in C^\infty(X) \), we denote its Cauchy data up to order \( m - 1 \) at \( \partial X \) by
\[ \gamma u = (\gamma_0 u, \gamma_1 u, \ldots, \gamma_{m-1} u) := (u, D_n u, \ldots, D_n^{m-1} u)|_{\partial X} \in C^\infty(\partial X)^m, \] (1.3)
where we denote by \( D_n \in \mathcal{V}(\mathbb{R}^n) \) a vector field transversal to \( \partial X \) (e.g. a smooth extension of the outward pointing normal vector field at \( \partial X \)). In order to encode boundary conditions, we are given boundary differential operators
\[ B_j = (B_{j,0}, B_{j,1}, \ldots, B_{j,m-1}), \quad B_{j,l} \in \text{Diff}^{s_{j,l}}(\partial X), \] (1.4)
and combine them into a single operator
\[ B = (B_1, \ldots, B_J). \] (1.5)
Given \( f \in C^\infty(X) \) and \( g_j \in C^\infty(\partial X), \ j = 1, \ldots, J \), we wish to solve the boundary value problem (BVP)
\[ \begin{cases}
Pu = f & \text{in } X, \\
B(\gamma u) = h \in C^\infty(\partial X)^J & \text{on } \partial X.
\end{cases} \] (1.6)
Thus, the \( j \)-th boundary condition reads
\[ B_j(\gamma u) = \sum_{l=0}^{m-1} B_{j,l}((D_n^l u)|_{\partial X}) = h_j. \] (1.7)
(This involves at most \( m - 1 \) transversal derivatives of \( u \) at \( \partial X \), but we do not need to restrict the number \( s_{j,l} \) of tangential derivatives.)
Example 1.1 (Dirichlet and Neumann problems). Let \( P = \Delta_{\mathbb{R}^n} \) denote the Laplace operator, and let \( X \subset \mathbb{R}^n \) denote any smoothly bounded domain. Take \( D_n \) to be the outward pointing normal vector field at \( \partial X \). Then the Dirichlet problem is of the form (1.6) for \( J = 1 \) and \( B_1 = (B_{1,0}, B_{1,1}) = (I, 0) \) (with \( I \) the identity operator), whereas the Neumann problem is of this form for \( J = 1 \) and \( B_1 = (B_{1,0}, B_{1,1}) = (0, I) \).

Calderón’s rough idea, in the case \( f = 0 \), is the following. Consider the space
\[
\mathcal{C}(P) := \{ \gamma u : u \in C^\infty(X), \ Pu = 0 \} \subset C^\infty(\partial X)^m
\]
of all Cauchy data of homogeneous solutions of \( P \). This is a closed subspace. We need to study the restriction \( B|_{\mathcal{C}(P)} \), and indeed would ideally like \( B|_{\mathcal{C}(P)} : \mathcal{C}(P) \to C^\infty(\partial X)^J \) to be an isomorphism, or at least Fredholm. The strategy for checking this is as follows:

1. We shall define a projection map \( \mathcal{C} : C^\infty(\partial X)^m \to C^\infty(\partial X)^m \) with range \( \mathcal{C}(P) \), or at least we shall, roughly, almost do this (i.e. up to smoothing errors). We will show that ‘the’ Calderón projector \( \mathcal{C} \) is a matrix of ps.d.o.s, and its symbol \( \sigma(\mathcal{C}) \), valued in \( \text{End}(\mathbb{C}^m) \), is idempotent, and can be computed explicitly in terms of the symbol of \( P \).

2. The precise formulation of the Fredholm property of \( B|_{\mathcal{C}(P)} \) is then that the symbol \( \sigma(B) \) is an isomorphism \( \sigma(B) : \text{ran} \sigma(\mathcal{C}) \subset \mathbb{C}^m \to \mathbb{C}^J \).

2. Extended problem

We denote by \( \tilde{P} \in \text{Diff}^m(\mathbb{R}^n) \) an elliptic extension of \( P \), i.e. \( \tilde{P}u|_{X^0} = Pu \) for \( u \in C^\infty_c(X^0) \).

Exercise 2.1. Show that such an extension exists.

We immediately abuse notation and write \( P \) instead of \( \tilde{P} \) simply. We work in a collar neighborhood \( \partial X \times [0,1) \) of \( \partial X \) and denote the coordinate on \( [0,1) \) by \( x_n \), so \( D_n = i^{-1}\partial_{x_n} \); here \( x_n > 0 \) in the interior of \( X^0 \). We then write
\[
P = \sum_{j=0}^{m} P_j D_n^j, \quad P_j \in \text{Diff}^{m-j}(\partial X).
\]

We denote by \( p \in S^m_{\text{hom}}(T^*\mathbb{R}^n) \) and \( p_j \in S^{m-j}_{\text{hom}}(T^*\partial X) \) the principal symbols of \( P \) and \( P_j \).

Given \( u \in C^\infty(X) \), denote by
\[
u^0(x) = \begin{cases} u(x), & x \in X, \\ 0, & x \in \mathbb{R}^n \setminus X \end{cases}
\]
the extension of \( u \) by 0. We then have:

Lemma 2.2 (Jump formula). For \( u \in C^\infty(X) \),
\[
Pu^0 = (Pu)^0 + P^\epsilon \gamma u,
\]
where for \( U = (U_0, \ldots, U_{m-1}) \in C^\infty(\partial X) \) we define
\[
P^\epsilon U := i^{-1} \sum_{j=0}^{m-1} \sum_{k=0}^{j} (P_{j+1}U_{j-k}) \otimes D_n^k \delta = i^{-1} \sum_{l=0}^{m-1} \sum_{j=0}^{m-1-l} (P_{j+l+1}U_l) \otimes D_n^l \delta,
\]
where \( \delta = \delta(x_n) \).
Proof. The term $P_0$ in (2.1) does not give any contribution. Next, note that $D_n u^0 = i^{-1} (u|_{\partial X}) \otimes \delta$. Induction with respect to $j$ gives

$$D_n^{j+1} u^0 = (D_n^{j+1} u^0) + i^{-1} \sum_{k=0}^j \gamma_{j-k} u \otimes D_n^k \delta. \quad (2.5)$$

Let $G \in \Psi^{-m}(\mathbb{R}^n)$ be an elliptic parametrix of $P$, so

$$PG = I + R', \quad GP = I + R, \quad R, R' \in \Psi^{-\infty}(\mathbb{R}^n). \quad (2.6)$$

Applying $G$ to (2.3), we obtain

$$u^0 + Ru^0 = G(Pu)^0 + GP\gamma u. \quad (2.7)$$

In particular, if $f = Pu = 0$, and if $R = 0$, we would formally obtain

$$\gamma u^0 = \gamma GP\gamma u. \quad (2.8)$$

Note that the Cauchy data of $Ra^0 \in C^\infty(\mathbb{R}^n)$ are certainly well-defined; on the other hand, while we certainly have $GP\gamma u \in C^\infty(X^\circ)$ due to the pseudolocality of the ps.d.o. $G$, it is not clear why its Cauchy data should be well-defined. We pause to explain the crucial input.

2.1. Pseudodifferential operators satisfying the transmission condition. We work on $\mathbb{R}^{n-1} \times \mathbb{R}$ with coordinates denoted $x = (x', x_n)$. Let $\mu \in \mathbb{C}$ and $A \in \Psi^\mu_{cl}(\mathbb{R}^n)$, and write the asymptotic expansion of its left symbol $a \in S^\mu_{cl}(T^*\mathbb{R}^n)$ into homogeneous terms as

$$a(x, \xi) \sim \sum_{j \in \mathbb{N}_0} a_j(x, \xi), \quad a_j \in S^{\mu-j}_{\text{hom}}(\mathbb{R}^n \setminus \{0\}). \quad (2.9)$$

**Definition 2.3** (Transmission condition). The operator $A$ satisfies the transmission condition with respect to the half space $x_n \geq 0$ if and only if

$$\partial_x^\alpha \partial_\xi^\beta a_j(x', 0, 0, -1) = e^{i\pi(\mu-j-|\beta|)} \partial_x^\alpha \partial_\xi^\beta a_j(x', 0, 0, 1). \quad (2.10)$$

This condition is satisfied when $a_j$ is a rational function of $\xi$, of degree $\mu - j$ where $\mu \in \mathbb{Z}$. It suffices to verify (2.10) for $\alpha = \beta = 0$, in which case (2.10) is the special case $(x', x_n, \xi', \xi_n) = (x', 0, 0, 1)$ of the statement that $a_j(x, -\xi) = (-1)^{\mu-j} a_j(x, \xi)$.

**Proposition 2.4** (Smooth extension). Suppose $A \in \Psi^\mu_{cl}(\mathbb{R}^n)$ satisfies the transmission condition. Let $U \in C^\infty(\mathbb{R}^{n-1})$ and $u(x', x_n) = U(x') \otimes \delta(x_n)$. Then $Au(x', x_n)$ extends smoothly from $x_n > 0$ to $\{x_n \geq 0\}$. Its boundary value is

$$\lim_{x_n \to 0^+} Au(x', x_n) = BU(x'), \quad (2.11)$$

where $B \in \Psi^{\mu+1}_{cl}(\mathbb{R}^{n-1})$ has symbol

$$b(x', \xi') \sim \sum_{j \in \mathbb{N}_0} b_j(x', \xi'), \quad b_j(x', \xi') = (2\pi)^{-1} \int^+ a_j(x', 0, \xi', \xi_n) \, d\xi_n, \quad (2.12)$$

with $\int^+$ defined by Lemma 2.5 below.
Lemma 2.5 (Integral). Suppose $q: \mathbb{R} \to \mathbb{C}$ is continuous, and assume that there exist $R > 0$ and an analytic function $Q: \Omega_R := \{ \zeta \in \mathbb{C}: \Im \zeta \geq 0, |\zeta| \geq R \}$ so that $|Q(\zeta)| \leq C|\zeta|^N$ for some $C, N,$ and $|q(z) - Q(z)| \leq C'|z|^{-2}$ when $\mathbb{R} \ni z \to \pm \infty$. Then

$$\int_+^+ q(z) \, dz := \int_{(\infty,-R) \cup (R,\infty)} (q(z) - Q(z)) \, dz + \int_{-R}^R q(z) \, dz - \int_R^R Q(z) \, dz \quad (2.13)$$

is independent of the choice of $Q$; here $\gamma_R$ is the upper semicircle of radius $R$, traversed clockwise. Moreover, if $F(z,y)$ is analytic in $\{z \in \mathbb{C}, \Im z \geq 0\}$ for $0 \leq y \leq 1$, and $F$ is bounded and continuous, then $\int_+^+ q(z) F(z,y) \, dz \in C^0([0,1],y)$.

Proof. It suffices to consider the case $q = 0$. The maximum principle implies for any $\epsilon > 0$ that

$$\sup_{\Omega_R} |z^2 Q(z)(1 - i\epsilon z)^{-N-3}| \leq \sup_{\partial \Omega_R} |z^2 Q(z)|, \quad (2.14)$$

since $|1 - i\epsilon z| \geq 1 + \epsilon |\Im z| \geq 1$ on $\Omega_R$. Letting $\epsilon \to 0$ implies that $|Q(z)| \leq C|z|^{-2}$ in $\Omega_R$, and hence $\int_{\partial \Omega_R} Q(z) \, dz = 0$ by analyticity. This proves that (2.13) is well-defined.

The final statement is clear, since we may take $Q(z) F(z,y)$ as the analytic extension of $q(z) F(z,y)$ from $\{z \in \mathbb{R}: |z| \geq R\}$.

Example 2.6. Suppose $q$ is a rational function of $z$ which is regular on $\mathbb{R}$. Consider the Laurent expansion of $q(w^{-1})$ at $w = 0$, given by $q(w^{-1}) = \sum_{j \geq -J} q_j w^{-j}$; we may then take $Q(z) := \sum_{j=-J}^1 q_j z^{-j}$ to be the truncated sum, which satisfies the conditions of Lemma 2.5. Upon letting $R \to \infty$ in (2.13), one thus finds that $\int_+^+ q(z) \, dz$ is the limit of $\int_{\gamma_R}^* q(z) \, dz$ as $R \to \infty$ where $\gamma_R^*$ is the concatenation of $[-R,R]$ and the upper semicircle of radius $R$, traversed counterclockwise; thus, by the residue theorem,

$$\int_+^+ q(z) \, dz = 2\pi i \sum_{j=1}^J \text{Res}_{z=z_j} q(z), \quad (2.15)$$

where $z_1, \ldots, z_J \in \{z \in \mathbb{C}: \Im z > 0\}$ is the list of all roots of $q$ in the upper half plane.

Proof of Proposition 2.4. It suffices to consider the case that the full symbol $a$ is independent of $x_n$; indeed, for $k \in \mathbb{N}_0$, one can integrate by parts in the oscillatory integral

$$\text{Op}(a^{x_n}_{x_n})u(x',x_n) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}} e^{i(x' \xi' + x_n \xi_n)} (-D_{\xi_n}^k) a(x',x_n,\xi',\xi_n) \hat{U}(\xi') \, d\xi_n \, d\xi', \quad (2.16)$$

thus effectively reducing the symbolic order of $a$; but for any fixed finite amount $k \in \mathbb{N}_0$ of regularity, $Au$ is $C^k$ down to $x_n = 0$ when the order of $A$ is sufficiently negative.

To prove the Proposition then, we approximate $\delta$ by $\phi_\epsilon(x_n) := \epsilon^{-1} \phi(\epsilon^{-1} x_n)$ where $\phi \in \mathcal{C}_c^\infty((-1,1))$ is nonnegative and satisfies $\int \phi(x_n) \, dx_n = 1$. Thus, $Au$ is the distributional limit of $Au_\epsilon$ as $\epsilon \searrow 0$, where $u_\epsilon(x',x_n) = U(x') \phi_\epsilon(x_n)$. We have

$$A u_\epsilon(x',x_n) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}} e^{i(x' \xi' + x_n \xi_n)} a(x',x_n,\xi',\xi_n) \phi_\epsilon(\epsilon \xi_n) \hat{U}(\xi') \, d\xi_n \, d\xi'. \quad (2.17)$$

Consider now a term $a_j(x,\xi',\xi_n)$ in (2.9), which we may assume to be independent of $x_n$. By homogeneity, we have

$$a_j(x',0,\xi',\xi_n) = |\xi_n|^{\mu - j} a_j(x',0,\xi'/|\xi_n|,\xi_n/|\xi_n|). \quad (2.18)$$
For fixed $\xi' \in \mathbb{R}^{n-1}$, the second factor on the right can be expanded in Taylor series around $\xi'/|\xi_n| = 0$. The transmission condition (2.10) shows that the term of order $\alpha' \in \mathbb{N}^{n-1}$ is given by $\xi_n^{\mu-j-|\alpha'|} (\xi')^{\mu'} \partial^{\alpha'}_{\xi} a_j(x', 0, 0, 1)$ and thus extends analytically to the punctured upper complex plane in $\xi_n$.

Returning to the full symbol $a$, we have

$$|a(x', 0, \xi', \xi_n) - \sum_{j+|\alpha'| < 2+\Re \mu} \xi_n^{\mu-j-|\alpha'|} (\xi')^{\mu'} \partial^{\alpha'}_{\xi} a_j(x', 0, 0, 1)| \leq C|\xi_n|^{-2}, |\xi_n| \geq 1. \quad (2.19)$$

We can thus apply Lemma 2.5 with $q(\xi_n) = a(x, \xi', \xi_n)$ for fixed $x, \xi'$, and with $F(\xi_n, \epsilon) = e^{ix_n \xi_n} \hat{\phi}(\epsilon \xi_n)$, which satisfies $|F(\xi_n, \epsilon)| \leq 1$ for $\Im \xi_n \geq 0$ provided that $x_n \geq \epsilon$. (This uses that $\phi$ has support contained in $(-1, 1)$, whence $|\hat{\phi}(\epsilon \xi_n)| \leq e^{\epsilon \Im \xi_n}$.) This gives

$$\int e^{ix_n \xi_n} a(x, \xi', \xi_n) \hat{\phi}(\epsilon \xi_n) d\xi_n \to \int e^{ix_n \xi_n} a(x, \xi', \xi_n) d\xi_n. \quad (2.20)$$

Indeed, for $\epsilon > 0$, the function $\hat{\phi}(\epsilon \xi_n)$ is Schwartz when $\Re \xi_n \to \pm \infty$, hence the Riemann integral $\int$ on the left is equal to $\int^+$ (since in Lemma 2.5 we may take $Q \equiv 0$ simply), and the existence and expression for the limit follows from the Lemma as well. We also note that the integral on the left has a bound by $C(|\xi'|^N \xi_n^n$ for some $C, N$ which are independent of $\epsilon$ and $x_n$.

We may now take $x_n \searrow 0$ after taking the $\epsilon \searrow 0$ limit of (2.17), and we conclude that

$$Au(x', x_n) \xrightarrow{x_n \searrow 0} BU(x'), \quad (2.21)$$

where the symbol of $B = \text{Op}(b)$ is given by

$$b(x', \xi') = (2\pi)^{-1} \int a(x', 0, \xi', \xi_n) d\xi_n. \quad (2.22)$$

It is easy to show that $b$ is in fact the asymptotic sum (2.12). Indeed, write

$$a(x', 0, \xi', \xi_n) = \sum_{j=0}^{N-1} a_j(x', 0, \xi', \xi_n) + r_N(x', \xi', \xi_n)$$

where $r_N = r_N(x', \xi', \xi_n)$ is a symbol of order $\Re \mu - N$ in $(\xi', \xi_n)$ when $|\xi'| > 1$ (thus working in the region where the $a_j$ are smooth). Note first that due to the degree $\mu - j$ homogeneity of $a_j$, the symbol $b_j$ in (2.12) is homogeneous of degree $\mu - j + 1$, as follows by a simple change of coordinates in $\xi_n$. Finally, in the computation of (2.13) for $q = r_N$, we can simply take $Q \equiv 0$, and it remains to note that

$$\left| \int r_N(x', \xi', \xi_n) d\xi_n \right| \leq C \int (1 + |\xi'| + |\xi_n|)^{\Re \mu - N} d\xi_n \leq C'(1 + |\xi'|)^{\Re \mu - N + 1}, \quad (2.23)$$

similarly for symbolic derivatives. The proof is complete.

\[\square\]

2.2. Calderón projector. Since $P$ is a differential operator, its parametrix $G$ satisfies the transmission condition of Definition 2.3, as each term in the symbol expansion is a rational function of $\xi$, as is easily seen by an inductive argument starting with the principal symbol $1/p(x, \xi)$. Therefore, equation (2.8) is well-defined by Proposition 2.4 (see below for details). More precisely, the Cauchy data of $GP^c \gamma u$ are defined as limits of Cauchy data at $x_n = \epsilon$ as $\epsilon \searrow 0$. 

5
Definition 2.7 (Calderón projector). The operator
\[ C := \gamma GP^c : \mathcal{C}^\infty(\partial X)^m \to \mathcal{C}^\infty(\partial X)^m \] (2.24)
is called ‘the’ Calderón projector; it only depends on the choice of parametrix \( G \), and is thus well-defined modulo smoothing operators.

Writing \( C = (C_{kl})_{k,l=0,\ldots,m-1} \), note that in view of the formula (2.4) for \( P^c \), we have
\[ C_{kl}U_t = \gamma_k \left( \sum_{j=0}^{m-1-l} G \left( P_{j+t+1} U_t \otimes D^n_j \delta \right) \right). \] (2.25)

Application of Proposition 2.4 with \( A \) satisfies \( u \Ψ \) (Properties of \( \xi \) by Example 2.6, this is the sum of the residues of the integrand in \( \text{Im} \xi_n > 0 \)).

Proposition 2.8 (Properties of \( C \)). The operator \( C \) is an approximate projection: \( C^2 - C \in \Psi^{-\infty}(\partial X; C^m) \) has smooth Schwartz kernel.

Proof. Let \( U \in \mathcal{C}^\infty(\partial X)^m \) and put \( u = GP^cU \), so \( u \in \mathcal{C}^\infty(X) \) by Proposition 2.4. Moreover, \( u \) satisfies
\[ Pu = (I + R')P^cU = P^cU + R'P^cU. \] (2.28)
Upon restriction to \( X^\circ \), the first term on the right goes away, and thus \( Pu = R'P^cU \) in \( X^\circ \), which is smooth up to \( \partial X \). By definition, \( \gamma u = CU \). We aim to compute \( C(\gamma u) = \gamma GP^c(\gamma u) \).

Consider again the formula (2.7) recovering \( \gamma u = u \), which here reads
\[ u^0 + Ru^0 = G(Pu)^0 + GP^c\gamma u = G(R'P^cU)^0 + GP^cCU. \] (2.29)
Taking Cauchy data from \( x_n > 0 \) gives
\[ CU + \gamma R(GP^cU)^0 = \gamma GP(U')^0 + C^2U. \] (2.30)
Since \( R' \) is smoothing and \( G \) satisfies the transmission condition, the operator \( U \mapsto \gamma G(RP^cU)^0 \) is smoothing.

We claim that also \( U \mapsto \gamma R(GP^cU)^0 \) is smoothing; this is best done via duality, namely for \( \phi \in \mathcal{C}^\infty(X) \) and, initially, \( U \in \mathcal{C}^\infty(\partial X)^m \), we have
\[ \langle (GP^cU)^0, \phi \rangle = \langle P^cU, G^s(\phi^0) \rangle. \] (2.31)
This is proved by first integrating only over a slightly smaller subdomain \( X_\epsilon \subset X^\circ \), whence the superscript 0 can be moved to \( \phi \) (now in the sense of extension by 0 outside of \( X_\epsilon \)), and one integrate by parts on \( X_\epsilon \); then one may let \( \epsilon \searrow 0 \), noting that \( G^s(\phi^0) \in \mathcal{C}^\infty(X) \) since also \( G^s \) satisfies the transmission condition. But now the right hand side of (2.31) is then also defined for \( U \in \mathcal{G}'(\partial X)^m \), which shows that \( (GP^cU)^0 \in \mathcal{G}'(\mathbb{R}^n) \) is well-defined as a distribution with support in \( X \). Since \( R \) is smoothing, we are done.  

\footnote{It is reassuring to look at this formula in the case that \( R = 0 \) and \( R' = 0 \).}
As already indicated in the footnote, the case \( R = R' = 0 \) is easier to parse. Note that this can be arranged in the setting \( P = \Delta \) considered in Example 1.1, as long as we are willing to take \( G \) to lie slightly outside of the calculus \( \Psi^{-2}(\mathbb{R}^n) \) defined in the lecture; namely, taking \( G \) to be convolution by \((4\pi r)^{-1}\) provides an exact inverse. (The weak off-diagonal decay of \( G \) is mostly harmless for our purposes of course, as we are working on a compact domain \( X \).)

We already have a formula (2.27) for the principal symbol of \( C \); we proceed to give a more conceptual description for it:

**Proposition 2.9** (Principal symbol of \( C \)). For \((x', \xi') \in T^* \partial X \setminus o\), consider the ODE

\[
\gamma U = (U(0), \ldots, D_n^{m-1}U(0)) \subset \mathbb{C}^m. \tag{2.32}
\]

Denote by \( M^\pm(x', \xi') \subset \mathbb{C}^m \) the subspace of solutions which are exponentially decreasing on \( \mathbb{R}_\pm = \pm (0, \infty) \). Then the principal symbol \( c(x', \xi') \colon \mathbb{C}^m \to \mathbb{C}^m \) of \( C \) is equal to the projection to \( M^+(x', \xi') \cap M^-(x', \xi') \) along \( M^-(x', \xi') \).

**Proof.** Since for nonzero \( \xi' \), the polynomial \( p(x', 0, \xi', \xi_n) = 0 \) has no real roots \( \xi_n \) due to the ellipticity of \( p \), solutions of \( p(x', 0, \xi', D_n)U(x_n) = 0 \) either grow exponentially or decay exponentially as \( x_n \to \infty \), likewise as \( x_n \to -\infty \). Therefore, \( M^+(x', \xi') \cap M^-(x', \xi') = 0 \) and \( M^+(x', \xi') \oplus M^-(x', \xi') = \mathbb{C}^m \).

Let now \( U \in M^+(x', \xi') \), identified with the solution \( U(x_n) \) of the ODE as well as its Cauchy data \((U_0, \ldots, U_{m-1})\). Define \( V(x_n) \) to be equal to \( U(x_n) \) for \( x_n > 0 \), and \( V(x_n) = 0 \) for \( x_n < 0 \). Then the jump formula (2.4) gives

\[
P(x', 0, \xi', D_n)V(x_n) = i^{-1} \sum_{l+j \leq m-1} p_{j+l+1}(x', \xi')U_lD_n^j \delta. \tag{2.33}
\]

On the other hand, using the Fourier transform in \( x_n \), we can write down a formula for the (unique tempered) solution \( V \) of this equation, to wit

\[
V(x_n) = (2\pi i)^{-1} \int e^{ix_n \xi_n} p(x', 0, \xi', \xi_n)^{-1} \sum_{l+j \leq m-1} p_{j+l+1}(x', \xi')U_l \xi_n^j \, d\xi_n. \tag{2.34}
\]

For \( x_n > 0 \), we can close the integration contour into the upper half plane, hence replace \( \int \) by \( \int^+ \). Taking Cauchy data at \( x_n = 0 \) (from the side \( x_n > 0 \)), and noting that the symbol of \( D_n^k \) is \( \xi_n^k \), we thus find from (2.27) that

\[
\gamma_k V = \sum_{l=0}^{m-1} \sigma(C_{kl})(x', \xi')U_l, \tag{2.35}
\]

so \( U = \gamma V = c(x', \xi')U \). Therefore, \( M^+(x', \xi') \subset \text{ran} \, c(x', \xi') \).

If on the other hand \( U \in M^-(x', \xi') \), the distribution \( V(x_n) \) defined by the same expression (2.34) defines the unique tempered solution of (2.33); on the other hand, the extension by 0 of \(-U|_{\mathbb{R}_-}\) satisfies the same ODE (the minus sign due to extending by 0 to \( \mathbb{R}_+ \) rather than \( \mathbb{R}_- \)). Thus, \( V \) vanishes on \( \mathbb{R}_+ \), and therefore, the Cauchy data of \( V \) from \( x_n > 0 \) vanish, i.e. \( c(x', \xi')U = 0 \). This shows that \( M^-(x', \xi') \subset \ker \, c(x', \xi') \). By dimension counting, we must have \( M^+(x', \xi') = \text{ran} \, c(x', \xi') \) and \( M^-(x', \xi') = \ker \, c(x', \xi') \), finishing the proof. \( \square \)
Example 2.10 (Laplacian). Consider the Laplacian $\Delta$ on a compact smoothly bounded domain $X$; for the calculation of the principal symbol of the Calderón projector, we may simply work with the (noncompact) model $X = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : x_n \geq 0\}$. We have $p(x', 0, \xi', \xi_n) = |\xi'|^2 + \xi_n^2$, so the solutions of $p(x', 0, \xi', D_{x_n})U = 0$ are given by

$$U(x', \xi'; x_n) = U_+ e^{-|\xi'| x_n} + U_- e^{i|\xi'| x_n},$$  

(2.36)

the first summand decaying exponentially as $x_n \to \infty$, the second as $x_n \to -\infty$. The Cauchy data are

$$\gamma U = (U_+ + U_-) = (\gamma_0, \gamma_1),$$  

(2.37)

from which we can conversely recover $U^\pm$ by the formula

$$(U_+, U_-) = \left( \frac{1}{2}(\gamma_0 - i|\xi'|^{-1}\gamma_1), \frac{1}{2}(\gamma_0 + i|\xi'|^{-1}\gamma_1) \right).$$  

(2.38)

Therefore, by computing $(\gamma_0, \gamma_1) \mapsto (U_+, U_-) \mapsto (U_+, 0) \mapsto \gamma U$ (the middle map projecting to the solution which decays as $x_n \to \infty$), we find

$$\sigma(C) = \left( \begin{array}{cc} \frac{1}{1-i|\xi'|} & \frac{1}{1-i|\xi'|} \\ -\frac{i}{2} \frac{|\xi'|^{-1}}{1-i|\xi'|} & \frac{i}{2} \frac{|\xi'|^{-1}}{1-i|\xi'|} \end{array} \right) = \left( \begin{array}{cc} \frac{1}{2} & -\frac{i}{2} |\xi'|^{-1} \\ -\frac{i}{2} |\xi'|^{-1} & \frac{1}{2} \end{array} \right).$$  

(2.39)

Thus, for instance, if we have a solution $u \in C^\infty(X)$ of the Dirichlet problem

$$\Delta u = 0, \quad u|_{\partial X} = h,$$  

(2.40)

then $Cu = \gamma u + R\gamma u$ where $R$ is smoothing. Write $\gamma u = (h, -i^{-1}\Delta h)$, where $\Lambda$ is the Dirichlet-to-Neumann map which computes the outward pointing normal derivative (hence the minus sign) of $u$ given its Dirichlet datum $h$. For some operator $S \in \Psi^1(\partial X)$ with principal symbol $|\xi'|$, we thus have $\frac{i}{2}Sh - \frac{1}{2}i^{-1}\Delta h = -i^{-1}\Delta h + Rh$ (from the second line of (2.39)), hence

$$\Lambda = S + 2iR \in \Psi^1(\partial X).$$  

(2.41)

Thus, the Dirichlet-to-Neumann map is an elliptic ps.d.o. with principal symbol $|\xi'|$. (An easy integration by parts moreover shows that it is self-adjoint and positive semidefinite.)

2.3. **Reduction to the boundary; ellipticity.** Consider again the BVP (1.6). Write equation (2.7) as

$$w^0 - GP^\epsilon\gamma u = G(Pu)^0 - Ru^0,$$  

(2.42)

and apply $\gamma$ to this (with $U = \gamma u$). We then see that, up to a smoothing error, it only remains to solve

$$(I - C)U = \gamma Gf^0, \quad BU = h,$$  

(2.43)

i.e. a system of pseudodifferential equations on $\partial X$. If one reduces to $f \equiv 0$, one wants to find $U$ satisfying $U = CU$ (up to smoothing errors) and $BU = h$.

Since $Q, B$ are (matrices of) pseudodifferential operators, the solvability and uniqueness properties of the system (2.43) can be read off from symbolic properties of $Q, B$. Let us rewrite (and generalize) the boundary conditions (1.7) as

$$B_j U = \sum_{k=0}^{m-1} B_{jk} U_k = h_j, \quad B_{jk} \in \Psi^{m_j-k}(\partial X),$$  

(2.44)

where $m_j = \max_{l=0,\ldots,m-1}(s_{jl} + l)$ is the total number of derivatives of $u$ involved in the computation of $B_j(\gamma u)$ in (1.7), and $U = \gamma u \in C^\infty(\partial X)^m$. 


Definition 2.11 (Elliptic boundary value problems). The boundary value problem (1.6) is elliptic if $P$ is elliptic and if moreover the boundary conditions are elliptic in the following sense: for each $x' \in \partial X$ and $\xi' \in T^*\partial X \setminus o$, the map

$$M^+(x',\xi') \ni U \mapsto \left( \sum_{k=0}^{m-1} b_{jk}(x',\xi') D^n_k U(0) \right)_{j=1,\ldots,J} \in C^J$$

(2.45)

is an isomorphism, where $M^+(x',\xi')$ is the set of all $U \in C^\infty(\mathbb{R})$ which satisfy the ODE $p(x',0,\xi',D_n)U(x_n) = 0$ and which are bounded on $\mathbb{R}_+$. (See also Proposition 2.9.)

Theorem 2.12 (Fredholm property). If the boundary value problem (1.6) is elliptic, then the map

$$ \tilde{H}^s(X^\circ) \ni u \mapsto (Pu, B_1(\gamma u), \ldots, B_J(\gamma u))$$

$$\in \tilde{H}^{s-m}(X) \oplus H^{s-m-\frac{1}{2}}(\partial X) \oplus \cdots \oplus H^{s-m-J-\frac{1}{2}}(\partial X)$$

(2.46)

is a Fredholm operator for any $s \geq m$.

Here, we write $\tilde{H}^s(X^\circ)$ for the space of restrictions to $X^\circ$ of elements of $H^s(\mathbb{R}^n)$. (These are so-called extendible distributions.) Note here that for $k \leq m-1$ and $u \in \tilde{H}^s(X^\circ)$, $s \geq m$, we have $D^n_k u \in H^s-k(X^\circ)$, which in view of $s-k > \frac{1}{2}$ thus has a well-defined restriction $\gamma_k u \in H^{s-k-\frac{1}{2}}(\partial X)$ to the hypersurface $\partial X$.

We shall not prove Theorem 2.12 here. Instead, we shall content ourselves with constructing a parametrix for the boundary problem (2.43), which is the main ‘fun’ microlocal bit. Retracing the arguments for the original BVP (1.6) allows one to construct a parametrix for the BVP, though the proof that one indeed obtains a two-sided parametrix is a bit technical; see [Hör07, §20.1] for details.

The main technical result regarding a system of the sort (2.43) is the following, stated with constant orders for notational simplicity:

Lemma 2.13. Let $Y$ be a compact manifold without boundary, and let $E, F \to Y$ denote complex vector bundles.\(^2\) Let $C \in \Psi^0(Y; E)$ with $C^2-C \in \Psi^{-\infty}$, and let $B \in \Psi^\mu(Y; E, F)$. Denote by $c, b$ the principal symbols of $C, B$.

1. If $b(y, \eta)|_{\text{ran } c(y, \eta)} : E_y \to F_y$ is surjective for all $(y, \eta) \in T^*Y \setminus o$, then there exists $S \in \Psi^{-\mu}(Y; F, E)$ so that

$$BS \equiv I, \quad CS \equiv S,$$

(2.47)

where ‘$\equiv$’ denotes equality modulo $\Psi^{-\infty}$.

2. If $b(y, \eta)|_{\text{ran } c(y, \eta)}$ is injective for all $(y, \eta) \in T^*Y \setminus o$, then there exist $S' \in \Psi^{-\mu}(Y; F, E)$ and $S'' \in \Psi^0(Y; E)$ so that

$$S'B + S'' \equiv I, \quad S''c \equiv 0.$$

(2.48)

3. If the conditions of both parts are satisfied, then $S, S', S''$ are uniquely determined modulo smoothing operators, and $S' \equiv S$.

Thus, roughly speaking (or passing to principal symbols), $S$ maps into the range of $C$ and provides a right inverse (i.e. solution operator) for $B$. In view of $S'B + S''(I-C) \equiv I$, thus $CS'B \equiv C$, the operator $S'$ recovers the piece of the argument of $B$ lying in the range

\(^2\)For us, $Y = \partial X$, $E = \mathbb{C}^m$ and $F = \mathbb{C}^J$. 

of $C$ from the output of $B$, whereas $S''$ recovers the piece in the range of the complementary projection $I - C$.

**Proof of Lemma 2.13.** Since $BC \in \Psi^\mu(Y; E, F)$ has surjective principal symbol, there exists $G \in \Psi^{-\mu}(Y; F, E)$ so that $BCG \equiv I$. Let then $S = CG$; then $BS \equiv I$, and $CS = C^2G \equiv CG = S$. This proves the first part.

For the second part, consider the case $\mu = 0$ for notational simplicity. The operator $B \oplus (I - C) \in \Psi^0(Y; E, E \oplus F)$. (One may shift the order of $B$ to 0 for notational simplicity.) By assumption, its principal symbol $b \oplus (I_E - c)$ is injective, hence we can find a left inverse $G' \oplus G''$, where (restoring orders) $G' \in \Psi^{-\mu}(Y; F, E)$ and $G'' \in \Psi^0(Y; E)$, and

$$G' B + G'' (I - C) \equiv I.$$  

(2.49)

We then set $S' = G'$ and $S'' = G'' (I - C)$; note that $S'' C = G'' (C^2 - C) \equiv 0$.

The proof of the final part is abstract nonsense: we have

$$S' \equiv S' BS \equiv (I - S'') S = S - S'' S \equiv S - S'' CS \equiv S.$$  

(2.50)

This gives $S \equiv S'$, and then $S''$ is of course uniquely defined modulo $\Psi^{-\infty}$ by (2.48). □

Restoring orders, we conclude that for our elliptic boundary value problem, there exist

$$S_{kj} \in \Psi^{k-m_j} (\partial X), \quad S'_{kj} \in \Psi^{k-l} (\partial X),$$  

(2.51)

so that for $S = S' = (S_{kj})$ and $S'' = (S''_{kj})$ we have (2.47)–(2.48). Consider again the ‘Green’s formula’ (2.7) and apply $\gamma$, then with $U = \gamma u$

$$(I - C) U + \gamma Ru^0 = \gamma Gf^0$$  

(2.52)

(which is (2.43) but with error terms); the boundary conditions read $BU = (B_j U)_{j=1,\ldots,J} = h = (h_j)_{j=1,\ldots,J}$. But then

$$I = SB + S'' (I - C) + \tilde{R}, \quad \tilde{R} \in \Psi^{-\infty}$$  

(2.53)

implies that

$$U = Sh + S'' (\gamma Gf^0 - \gamma Ru^0) + \tilde{R} U.$$  

(2.54)

Since $u^0$ and $U$ are not independent, we need to work on this a bit further: namely, using (2.7), we can write an approximate solution of our BVP as

$$u^0 = Gf^0 + G\psi U - Ru^0 = (I + G\psi S'' \gamma) Gf^0 + G\psi Sh + R'' u^0,$$  

(2.55)

where $R'' = G\psi (\tilde{R} \gamma u^0 - S'' \gamma Ru^0) - Ru^0$ is a smoothing operator in the sense that it maps $H^m (X^\circ) \to C^\infty (X)$. We thus want to conclude that

$$[(f, h) \mapsto u := (I + G\psi S'' \gamma) Gf^0 + G\psi Sh]_{X^\circ}$$  

(2.56)

is an approximate left inverse of (1.6). With some work one can show:

**Theorem 2.14.** For any $s \geq m$, the map (2.56) is a bounded linear map

$$\tilde{H}^{s-m}(X^\circ) \oplus H^{s-m - 1/2}(\partial X) \oplus \cdots \oplus H^{s-m - 1/2}(\partial X) \to \tilde{H}^s(X^\circ).$$  

(2.57)
It is an approximate left inverse of (1.6) in the sense that (2.55) is valid for any approximate (i.e. modulo $C^\infty$) solution $u$ of (1.6) with $R'' : \tilde{H}^m(X^\circ) \to C^\infty(X)$ being a smoothing operator. It is also an approximate right inverse of (1.6) in the sense that $Pu = f + K_1 f + K_2 h$ and $B(\gamma u) = h + K_3 f + K_4 h$ where

$$
\begin{pmatrix}
K_1 & K_2 \\
K_3 & K_4
\end{pmatrix} : L^2(X) \oplus \mathcal{D}'(\partial X; \mathbb{C}^J) \to C^\infty(X) \oplus C^\infty(\partial X; \mathbb{C}^J)
$$

is continuous.

This of course implies the Fredholm statement of Theorem 2.12.

REFERENCES


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