Do exercises 1–8. The bonus exercises 1–3 are optional.

Due: Thursday, Feb 21, 5 pm. Hand it in in person, or send your pdf to phintz@mit.edu.

Exercise 1. (1) Show that $\Delta + 1: \mathscr{E}'(\mathbb{R}^n) \to \mathscr{E}'(\mathbb{R}^n)$ is an isomorphism.

(2) Find a non-trivial solution $u \in C^\infty(\mathbb{R}^n)$ of $(\Delta + 1)u = 0$. Why does this not contradict the first part?

Exercise 2. (Sobolev embedding.) Let $s > n/2$.

(1) Prove that there exists a constant $C_s < \infty$ such that for $\phi \in \mathcal{S}(\mathbb{R}^n)$, the estimate
$$
\|\phi\|_{L^\infty(\mathbb{R}^n)} \leq C_s \|\phi\|_{H^s(\mathbb{R}^n)}.
$$
holds. Deduce that $H^s(\mathbb{R}^n) \subset C^0_b(\mathbb{R}^n)$.

(2) Show more generally that $H^s(\mathbb{R}^n) \subset C^k_b(\mathbb{R}^n)$ for $s > n/2 + k$.

(3) Prove that $\mathcal{S}(\mathbb{R}^n) = \bigcap_{r,s \in \mathbb{R}} \langle x \rangle^r H^s(\mathbb{R}^n)$.

Exercise 3. Show that the principal symbol $\sigma_m(A)$ of $A \in \text{Diff}^m(\mathbb{R}^n)$ captures the ‘high frequency behavior’ of $A$ in the following sense: for $x_0, \xi_0 \in \mathbb{R}^n$, we have
$$
\sigma_m(A)(x_0, \xi_0) = \lim_{\lambda \to \infty} \lambda^{-m}(e^{-i\lambda\xi_0} Ae^{i\lambda\xi_0})(x_0),
$$
where $e^{i\xi_0 \cdot}$ is the function $x \mapsto e^{i\xi_0 \cdot x}$.

Exercise 4. (1) Let $m \in \mathbb{R}$. Show that $\langle \xi \rangle^m \in S^m(\mathbb{R}^N)$.

(2) Let $\mu \in \mathbb{C}$. Show that $\langle \xi \rangle^\mu \in S^\mu_{cl}(\mathbb{R}^N)$.

Exercise 5. (1) Suppose $a \in S^m(\mathbb{R}^n; \mathbb{R}^N)$ satisfies $|a(x, \xi)| \geq c|\xi|^m$ for $|\xi| \geq C$ and some $c > 0$. Let $\chi \in S^0(\mathbb{R}^N)$ be identically 0 for $|\xi| \leq 2C$. Show that $\chi/a \in S^{-m}(\mathbb{R}^n; \mathbb{R}^N)$.

(2) If in addition $a$ and $\chi$ are classical symbols, show that $\chi/a$ is classical as well.

Exercise 6. (1) Let $f \in C^\infty(\mathbb{R})$. Show that if $a \in S^0(\mathbb{R}^n; \mathbb{R}^N)$, then also $f \circ a \in S^0(\mathbb{R}^n; \mathbb{R}^N)$.

(2) Show that if $a \in S^0(\mathbb{R}^n; \mathbb{R}^N)$ is elliptic and positive, then there exists $b \in S^0(\mathbb{R}^n; \mathbb{R}^N)$ such that $a - b^2 \in S^{-1}(\mathbb{R}^n; \mathbb{R}^N)$.

Exercise 7. Let $m \in \mathbb{N}_0$, and let $a \in S^m(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n)$ be a polynomial in the symbolic variable $\xi$. 

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1See the course website, https://math.mit.edu/~phintz/18.157-S19/, for homework policies.
(1) Show, starting from the definition as a limit of quantizations of residual symbols, that $\text{Op}(a) \in \text{Diff}^m(\mathbb{R}^n)$.  
(2) Prove that $\text{Op}((x - y)^\omega a) \in \text{Diff}^m(\mathbb{R}^n)$ (which in particular entails the boundedness of the coefficients). \textit{(Hint.} Compute its Schwartz kernel.\textit{)}

\textbf{Exercise 8.} Let $A \in \Psi^m(\mathbb{R}^n)$, and denote by $K$ its Schwartz kernel.

1. Give a direct proof that $K \in C^\infty((\mathbb{R}^n \times \mathbb{R}^n) \setminus \Delta)$, where $\Delta = \{ (x, x) : x \in \mathbb{R}^n \}$ is the diagonal. \textit{(Hint.} For $\phi, \psi \in C_c^\infty(\mathbb{R}^n)$ with $\text{supp} \psi \cap \text{supp} \psi = \emptyset$, rewrite the pairing $\langle A\phi, \psi \rangle$ for $A \in \Psi^{-\infty}(\mathbb{R}^n)$ using integrations by parts. Then use a density argument.\textit{)}

2. Prove that for every $\epsilon > 0$, $N \in \mathbb{R}$ there exists a constant $C$ such that
   $$|K(x, y)| \leq C|x - y|^{-N}, \quad |x - y| \geq \epsilon.$$

\textbf{Bonus exercise 1.} \textit{(Schwartz kernel theorem I.)} Let $K \in \mathscr{S}(\mathbb{R}^{n+m})$, and define the linear operator $O_K : \mathscr{S}(\mathbb{R}^m) \to \mathscr{S}(\mathbb{R}^n)$ by
   $$\langle O_K \phi, \psi \rangle := \langle K, \psi \otimes \phi \rangle, \quad \phi \in \mathscr{S}(\mathbb{R}^m), \psi \in \mathscr{S}(\mathbb{R}^n).$$

Suppose $O_K = 0$. Show that $K = 0$. \textit{(Hint.} Given $\phi \in \mathscr{S}(\mathbb{R}^{n+m})$, you need to show that $\langle K, \phi \rangle = 0$. You know that this is true when $\phi$ is a finite linear combination of exterior products $\psi_1 \otimes \psi_2$, $\psi_1 \in \mathscr{S}(\mathbb{R}^n), \psi_2 \in \mathscr{S}(\mathbb{R}^m)$. Try to use the Fourier transform, or Fourier series, to approximate $\phi$ by such linear combinations. It may help to first reduce to the case that $\text{supp} K$ is compact.\textit{)}

\textbf{Bonus exercise 2.} \textit{(Schwartz kernel theorem II.)} Let $A : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^{m})$ be continuous. Prove that there exists $K \in \mathscr{S}(\mathbb{R}^{n+m})$ such that $A = O_K$. You may follow these steps:

1. The continuity of $A$ is equivalent to the statement that for all $\psi \in \mathscr{S}(\mathbb{R}^n)$ there exists $N > 1$ such that $|\langle A\phi, \psi \rangle| \leq N \|\phi\|_N$ for all $\phi \in \mathscr{S}(\mathbb{R}^m)$.
2. There exist $N, M \in \mathbb{R}$ such that $A$ extends by continuity to a bounded operator
   $$A : \langle x \rangle^{-M} H^M(\mathbb{R}^m) \to \langle x \rangle^N H^{-N}(\mathbb{R}^n).$$

3. The operator
   $$A' := \langle D \rangle^{-N-n/2-1} \langle x \rangle^{-N} A \langle D \rangle^{-M-m/2-1} \langle x \rangle^{-M}$$
   is bounded from $H^{-m/2-1}(\mathbb{R}^m)$ to $C^0_b(\mathbb{R}^n)$
4. Evaluate $A' \delta_y$ for $y \in \mathbb{R}^m$ and deduce that $A'$ has a Schwartz kernel $K' \in C^0_b(\mathbb{R}^{n+m})$.  
5. By relating the Schwartz kernels of $A'$ and $A$, prove that $A = O_K$ for some $K \in \mathscr{S}(\mathbb{R}^{n+m})$.

\textbf{Bonus exercise 3.} Let $A : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ be a continuous linear operator, and suppose for all $u \in \mathscr{S}(\mathbb{R}^n)$, we have $\text{supp} Au \subseteq \text{supp} u$. Prove that $A$ is a differential operator. \textit{(Hint.} Show that the Schwartz kernel $K$ of $A$ has support in the diagonal $\{ x = y \}$. Then show/recall that distributions with support on a submanifold $S$ are locally finite linear combinations of (differentiated) $\delta$-distributions (with coefficients in $C^\infty(S)$) at $S$. Do not forget to prove that $A$ is a differential operator of finite order.)