The lci locus of the Hilbert scheme of points & the cotangent complex

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Abstract

We introduce the basic algebraic geometry background necessary to understand the main results of the work of Elmanto, Hoyois, Khan, Sosnilo, and Yakerson on motivic infinite loop spaces [5]. After recalling some facts about the Hilbert functor of points, we introduce *local complete intersection* (*lci*) and *syntomic* morphisms. We then give an overview of the cotangent complex and discuss the relationship between lci morphisms and the cotangent complex (in particular, Avramov's characterization of lci morphisms). We then introduce the lci locus of the Hilbert scheme of points and the Hilbert scheme of framed points, and prove that the lci locus of the Hilbert scheme of points is formally smooth.

Contents

1 Hilbert schemes

In this section we recall the Hilbert functor of points from last time as well as state the basic representability properties of the Hilbert functor of points.

1.1 Recollection ([STK, Tag 02K9])**.** A morphism of schemes $f\colon Y \to X$ is ***finite locally free*** if f is affine and $f_\star \mathcal{O}_Y$ is a finite locally free $\mathcal{O}_X$-module. This is equivalent to saying that f is finite, flat, and locally of finite presentation. If X is locally noetherian, the condition that f be locally of finite presentation is implied by the finiteness of f.

1.2 Definition. Let S be a scheme and $X \in \mathbf{Sch}_S$. The ***Hilbert functor of points*** is the functor $\mathrm{Hilb}^{\mathrm{fin}}(X/S)\colon \mathbf{Sch}_S^{\mathrm{op}} \to \mathbf{Set}$ defined by sending $Y \in \mathbf{Sch}_S$ to the set of closed subschemes $Z \subset X \times_S Y$ that are finite and locally free over Y.

1.3. The degree of a finite locally free morphism induces a decomposition

$$\mathrm{Hilb}^{\mathrm{fin}}(X/S) \simeq \coprod_{d \geq 0} \mathrm{Hilb}_d^{\mathrm{fin}}(X/S)$$

of product-preserving presheaves on $\mathbf{Sch}_S$.

We have the following representability properties of the Hilbert functor of points:

1.4 Theorem (see [5, Lemma 5.1.3])**.** *Let S be a scheme, $X \in \mathbf{Sch}_S$, and $d \geq 0$.*

(1.4.1) *If $X \to S$ is separated, then $\mathrm{Hilb}^{\mathrm{fin}}(X/S)$ is representable by a separated algebraic space over S, which is:*

- (1.4.1.1) *Locally of finite presentation if $X \to S$ is.*
- (1.4.1.2) *A scheme if every finite set of points of every fiber of $X \to S$ is contained in an affine open in X (e.g., $X \to S$ is locally quasi-projective).*

(1.4.2) *If $X \to S$ is finite presented and locally (resp., strongly) quasi-projective, then $\mathrm{Hilb}_d^{\mathrm{fin}}(X/S)$ is locally (resp., strongly) quasi-projective over S.*

Proof. For assertion (1.4.1), combine [13, Theorem 1.1; 16, Theorem 4.1; STK, Tag 0B9A]. Assertion (1.4.2) is [1, Corollaries 2.7 & 2.8]. □

1.5 Remark. Nitsure gives a nice introduction to Hilbert and Quot functors [12].

1.6 Theorem (Fogarty [6, Theorem 2.4 & Corollary 2.6])**.** *Let k be a field and X a smooth surface over k. Then the Hilbert scheme $\mathrm{Hilb}_d^{\mathrm{fin}}(X/k)$ is smooth of dimension $2d$ and birational to $\mathrm{Sym}^d(X)$.*

1.7. Although the Hilbert scheme of points of a smooth surface is smooth, Hilbert schemes of points of smooth schemes are generally *not* smooth. In this talk we'll describe an open subscheme that is smooth, by considering the locus of points 'cut out by the minimal number of equations'.

2 Local complete intersections via equations

In this section we define and give examples of *relative global complete intersections* and *local complete intersections*, which make precise the notion of 'being cut out by the minimal number of equations'.

Relative global complete intersections

2.1 Definition. A morphism of affine schemes $f\colon \operatorname{Spec}(S) \to \operatorname{Spec}(R)$ is a ***relative global complete intersection*** if there exists a presentation

$$S \cong R[x_1, \dots, x_n]/(f_1, \dots, f_c) ,$$

such that all of the nonempty fibers of f have dimension $n - c$.

2.2 Example. Let k by a field and $I \subset k[x]$ an ideal such that $k[x]/I$ is a 0-dimensional scheme (equivalently, $k[x]/I$ is finite-dimensional as a k-algebra). Since $k[x]$ is a principal ideal domain, $I = (f)$ for some polynomial $f \in k[x]$. Thus $k[x]/I$ is a relative global complete intersection over k.

2.3 Example. Let k be a field, and $e_1, \dots, e_n$ positive integers. Then the ring

$$k[x_1, \dots, x_n]/(x_1^{e_1}, \dots, x_n^{e_n})$$

is 0-dimensional (since (0) is the only prime), hence a relative global complete intersection over k.

The following commuative algebra lemmas give some basic facts about relative global complete intersections that we use repeatedly.

2.4 Lemma ([STK, Tag 00SV]). *Let R be a ring and $S \coloneqq R[x_1, \dots, x_n]/(f_1, \dots, f_c)$ a relative global complete intersection. Let $\mathfrak{p} \in \operatorname{Spec}(S)$, and write $\mathfrak{p}'$ for the corresponding prime of $R[x_1, \dots, x_n]$. Then:*

(2.4.1) *The sequence $f_1, \dots, f_c$ is a regular sequence in the local ring $R[x_1, \dots, x_n]_{\mathfrak{p}'}$.*

(2.4.2) *For each $1 \leq i \leq c$, the ring $R[x_1, \dots, x_n]_{\mathfrak{p}'}/(f_1, \dots, f_i)$ is flat over R.*

(2.4.3) *The ring S is flat over R.*

(2.4.4) *The conormal module $(f_1, \dots, f_c)/(f_1, \dots, f_c)^2$ is a free S-module with basis given by the classes of $f_1, \dots, f_c$.*

2.5 Lemma ([STK, Tag 07CF]). *Let R be a ring, and $I \subset R[x_1, \dots, x_n]$ a finitely generated ideal. If the conormal module I/I^2 is free over $R[x_1, \dots, x_n]/I$, then there exists a presentation*

$$R[x_1, \dots, x_n]/I \cong R[y_1, \dots, y_m]/(f_1, \dots, f_c)$$

such that $(f_1, \dots, f_c)/(f_1, \dots, f_c)^2$ is free with basis given by the classes of $f_1, \dots, f_c$. In this case, $R[x_1, \dots, x_n]/I$ is a relative global complete intersection over R.

The following lemma gives a method for producing relative global complete intersections. First we recall the conormal sequence, which we make repeated use of.

2.6 Recollection (conormal sequence [STK, Tag 01UZ]). Let $i: Z \hookrightarrow X$ be a closed immersion of S-schemes, with ideal sheaf I. The differential $\mathrm{d}: I \subset \mathcal{O}_X \to \Omega_{X/S}$ maps $I^2 \subset I$ to $I\Omega_{X/S}$, hence induces an $\mathcal{O}_X/I$-linear map $\mathrm{d}: I/I^2 \to \Omega_{X/S}/I\Omega_{X/S}$. That is, d pulls back to a map $\mathrm{d}: \mathcal{N}_i \to i^\star\Omega_{X/S}$ to a map from the conormal sheaf of i to $i^\star\Omega_{X/S}$. Moreover, we have a *conormal exact sequence*

$$\mathcal{N}_i \xrightarrow{\mathrm{d}} i^\star\Omega_{X/S} \to \Omega_{Z/S} \to 0 \,.$$

2.7 Lemma (see [STK, Tag 00ST]). *Let X be an affine scheme, $p: U \to \mathbf{A}^n_X$ an affine étale morphism, and $Z \subset U$ a closed subscheme cut out by c equations. If the nonempty fibers of $Z \to \mathbf{A}^n_X \to X$ have dimension $n - c$, then $Z \to X$ is a relative global complete intersection.*

Proof. Factor $p: U \to \mathbf{A}^n_X$ as

$$\begin{tikzcd} U \arrow[r, hook, "i"] \arrow[dr, "p"'] & \mathbf{A}^{m+n}_X \arrow[d] \\ & \mathbf{A}^n_X \,, \end{tikzcd}$$

where i is a closed immersion. Since p is étale, $\Omega_p \cong 0$, so the conormal sequence

$$0 \to \mathcal{N}_i \to i^\star(\Omega_{\mathbf{A}^{m+n}_X/\mathbf{A}^n_X}) \to \Omega_p \to 0$$

provides an isomorphism

$$\mathcal{N}_i \cong i^\star(\Omega_{\mathbf{A}^{m+n}_X/\mathbf{A}^n_X}) \cong \mathcal{O}_U^{\oplus m} \,.$$

Choose functions $f_1, \ldots, f_m$ on $\mathbf{A}^{m+n}_X$ lifting to generators of $\mathcal{N}_i \cong \mathcal{O}_U^{\oplus m}$. By Nakayama's lemma, there is a function h on $\mathbf{A}^{m+n}_X$ such that U is cut out by $f_1, \ldots, f_m$ in the localization $(\mathbf{A}^{m+n}_X)_h$ of $\mathbf{A}^{m+n}_X$ at h. But then U is cut out by the $m+1$ equations $f_1, \ldots, f_m$, and $hx_{n+m+1} - 1$ in $\mathbf{A}^{m+n+1}_X$. Hence Z is cut out by $c + m + 1$ equations in $\mathbf{A}^{m+n+1}_X$, so (by definition) $Z \to X$ is a relative global complete intersection. □

Local complete intersections

In order to introduce local complete intersections, we recall some preliminaries on regularity conditions for immersions of schemes.

2.8 Recollection (the Koszul complex). Let R be a ring and $r \in R$. The ***Koszul complex*** $\mathrm{Koz}(r)$ of r is the complex

$$0 \to R \xrightarrow{\cdot r} R \to 0$$

concentrated in degrees 0 and 1. Note that there is an augmentation $\mathrm{Koz}(r) \to R/(r)$. Given a sequence of elements $r_1, \ldots, r_n \in R$, the ***Koszul complex*** $\mathrm{Koz}(r_1, \ldots, r_n)$ of the sequence $r_1, \ldots, r_n$ is the tensor product of complexes

$$\mathrm{Koz}(r_1, \ldots, r_n) \coloneqq \mathrm{Koz}(r_1) \otimes_R \cdots \otimes_R \mathrm{Koz}(r_n) \,.$$

Hence there is an induced augmentation

$$\mathrm{Koz}(r_1, \dots, r_n) \to R/(r_1, \dots, r_n) \ .$$

Explicitly, $\mathrm{Koz}_p(r_1, \dots, r_n) = \Lambda^{p+1}(R^n)$ with differential given by

$$\mathrm{d}(e_{i_0} \wedge \cdots \wedge e_{i_p}) = \sum_{k=0}^{p} (-1)^k r_{i_k} e_{i_0} \wedge \cdots \wedge \hat{e}_{i_k} \wedge \cdots \wedge e_{i_p} \ .$$

2.9 Recollection (regularity conditions for ideals [STK, Tag 07CU])**.** Let R be a ring. A sequence of elements $r_1, \dots, r_n \in R$ is ***Koszul-regular*** if the Koszul complex $\mathrm{Koz}(r_1, \dots, r_n)$ is a resolution of $R/(r_1, \dots, r_n)$, i.e.,

$$\mathrm{H}_i(\mathrm{Koz}(r_1, \dots, r_n)) = 0 \quad \text{for } i \neq 0 \ .$$

A *regular sequence* is necessarily Koszul-regular. If R is a noetherian ring, then every Koszul-regular sequence is also regular; see [STK, Tag 063I].

2.10 Definition. Let X be a scheme and $I \subset \mathcal{O}_X$ an ideal sheaf. We say that I is ***regular*** if for every $x \in \mathrm{supp}(\mathcal{O}_X/I)$ there is an open neighborhood $U \subset X$ of x and a Koszul-regular sequence $f_1, \dots, f_n \in \mathcal{O}_X(U)$ such that $I|_U$ is generated by $f_1, \dots, f_n$.

An immersion $Z \hookrightarrow X$ is ***regular*** if there is an open subscheme $Z \subset U \subset X$ such that Z is closed in U and the ideal sheaf $I_{Z \subset U} \subset \mathcal{O}_U$ is regular.

2.11 Definition. A morphism of schemes $f: Y \to X$ is a ***local complete intersection*** (***lci***) if locally on X, f is the composite of a regular immersion followed by a smooth morphism. We say that f is ***syntomic*** if f is flat and lci.

2.12. Since the families in the definition of the Hilbert scheme are flat, we are really more intereseted in syntomic maps.

2.13 Remark. Mazur introduced the word 'syntomic' as built from the verb 'temnein' (i.e., 'to cut') and the prefix 'syn' meaning 'same' or 'equal' [14]. So, roughly, 'syntomic' means 'eqicut'.

2.14 Example. Let k be a field. The embedding $i: \mathrm{Spec}(k) \hookrightarrow \mathrm{Spec}(k[x])$ induced by the map $k[x] \to k$ sending x to 0 is regular. However, the pullback of i along the map $\mathrm{Spec}(k[x]/(x^2)) \hookrightarrow \mathrm{Spec}(k[x])$ induced by the quotient map $k[x] \to k[x]/(x^2)$ corresponds to the map $k[x]/(x^2) \to k$ killing x, which is not regular. In particular, the basechange of an lci map need *not* be lci.

2.15 Proposition ([SGA 6, Exposé VIII, Proposition 1.6])**.** *The basechange of an lci morphism along a flat morphism is lci.*

2.16 Proposition ([STK, Tag 01UI])**.** *Syntomic morphisms are stable under basechange.*

2.17. We want a more extrinsic characterization of relative global complete intersections. The characterization that we will explore is in terms of the *cotangent complex*.

3 Background on the cotangent complex

Motivation

Given a morphism of S-schemes

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr] & & X \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

we have an exact sequence

$$f^{\star}\Omega_g \to \Omega_{gf} \to \Omega_f \to 0\,, \tag{3.1}$$

of quasicoherent sheaves on Y, and this sequence generally does *not* extend to a short exact. As usual, we are interested in extending (3.1) to a long exact sequence

$$\begin{tikzcd} \cdots \arrow[r] & \mathrm{H}_1(f^{\star}\mathcal{L}_g) \arrow[r] & \mathrm{H}_1(\mathcal{L}_{gf}) \arrow[r] & \mathrm{H}_1(\mathcal{L}_f) \arrow[dll, rounded corners] & \\ & f^{\star}\Omega_g \arrow[r] & \Omega_{gf} \arrow[r] & \Omega_f \arrow[r] & 0\,, \end{tikzcd}$$

where $\mathcal{L}_f$ and $\mathcal{L}_{gf}$ are complexes of quasicoherent sheaves on Y, and $\mathcal{L}_g$ is a complex of quasicoherent sheaves on X. Since the sequence (3.1) *does* extend to a short exact sequence when f is smooth, we should also have $\mathcal{L}_f \simeq \Omega_f[0]$ if f is smooth. The complex $\mathcal{L}_f$ is what is called the ***cotangent complex*** of f.

3.2 Notation. For a scheme X, we write $\mathbf{QCoh}(X)$ for the *derived ∞-category* of quasicoherent sheaves on X.

3.3 Remark (functoriality of Kähler differentials)**.** The quasicoherent sheaf of Kähler differentials associated to a morphism of schemes has a slightly complicated functoriality. The cleanest way to say this is that Kähler differentials is a functor fitting into the diagram

$$\begin{tikzcd} & \mathbf{QCoh} \arrow[d, "p"] \\ \mathrm{Fun}(\Delta^1, \mathbf{Sch})^{\mathrm{op}} \arrow[ur, "\Omega_{(-)}[0]"] \arrow[r, "s"'] & \mathbf{Sch}^{\mathrm{op}}\,, \end{tikzcd}$$

where s is the source functor and p is the cocartesian fibration classified by the diagram $X \mapsto \mathbf{QCoh}(X)$. (See (3.5) for an explicit description of what this means.)

3.4 Remark (the cotangent complex for affine schemes)**.** Let R be a ring. Write $\mathbf{D}(R)$ for the derived ∞-category of R, $\mathbf{Alg}_R$ for the cateogry of R-algebras, and $\mathbf{Alg}_R^{\mathrm{sm}}$ for the category of smooth R-algebras. If we forget the S-module structure on the cotangent complex of an R-algebra S, then we can define the cotangent complex as the left Kan extension

$$\begin{tikzcd} \mathbf{Alg}_R^{\mathrm{sm}} \arrow[r, "\Omega_{-/R}[0]"] \arrow[d, hook] & \mathbf{D}(R) \\ \mathbf{Alg}_R \arrow[ur, dotted, "\mathcal{L}_{-/R}"'] & \end{tikzcd}$$

of Kähler differentials (concentrated in degree 0) on smooth R-algebras. If we want to include the full functoriality and structure of the cotangent complex to align with Remark 3.3, we need to take a more involved *relative* left Kan extension (see [HA, §7.3.2]).

Often the cotangent complex for affines is presented by by left Kan extending Kähler differentials from polynomial R-algebras (rather than smooth R-algebras, and then one shows that the cotangenet complex of a smooth R-algebra is given by Kähler differentials; see [3, Example 2.2; 11, Remark 2.25]. The latter fact shows that these two left Kan extensions agree.

Properties of the cotangent complex

The cotangent complex for non-affine schemes can be defined by extending the cotangent complex for affine schemes by enforcing Zariski descent. Below is a summary of the relevant functoriality and properties of the cotangent complex:

3.5. As in Remark 3.3, the cotangent complex is a section

$$\begin{tikzcd} & \mathbf{QCoh} \arrow[d, "p"] \\ \mathrm{Fun}(\Delta^1, \mathbf{Sch})^{\mathrm{op}} \arrow[ur, "\mathcal{L}_{(-)}"] \arrow[r, "s"'] & \mathbf{Sch}^{\mathrm{op}} . \end{tikzcd}$$

Explicitly, this means that for each scheme Y, the cotangent complex defines a functor

$$\mathcal{L}_{(-)} \colon (\mathbf{Sch}_{Y/})^{\mathrm{op}} \to \mathbf{QCoh}(Y) \, ,$$

and has the following additional functoriality: given a commutative square

$$\begin{tikzcd} Y' \arrow[r, "c"] \arrow[d, "f'"'] & Y \arrow[d, "f"] \\ X' \arrow[r] & X \, , \end{tikzcd} \tag{3.6}$$

there is a natural morphism $c^\star \mathcal{L}_f \to \mathcal{L}_{f'}$ in $\mathbf{QCoh}(Y')$. Moreover, the cotangent complex satisfies the following properties.

(3.5.1) For any morphism $f \colon Y \to X$, the cotangent complex $\mathcal{L}_f$ is a connective object of $\mathbf{QCoh}(Y)$ (with respect to the usual t-structure).

(3.5.2) If (3.6) is a pullback and Tor-*independent* in the sense that $\mathrm{Tor}^i_{\mathcal{O}_X}(\mathcal{O}_{X'}, \mathcal{O}_Y) = 0$ for $i > 0$, then $c^\star \mathcal{L}_f \xrightarrow{\sim} \mathcal{L}_{f'}$.

(3.5.3) *Fundamental fiber sequence:* Given composable morphisms $Y \xrightarrow{f} X \xrightarrow{g} S$, we have a natural fiber sequence

$$f^{*}\mathcal{L}_{g} \to \mathcal{L}_{gf} \to \mathcal{L}_{f}$$

in $\mathbf{QCoh}(Y)$.

(3.5.4) Every morphism $f\colon Y \to X$ induces a morphism $\mathcal{L}_f \to \Omega_f[0]$ which induces an isomorphism $\mathrm{H}_0(\mathcal{L}_f) \xrightarrow{\sim} \Omega_f[0]$. Moreover, if f is smooth, then $\mathcal{L}_f \xrightarrow{\sim} \Omega_f[0]$.

(3.5.5) If i is a closed immersion, there is a natural isomorphism $\mathrm{H}_1(\mathcal{L}_i) \cong \mathcal{N}_i$.

Without knowing anything more specific about the cotangent complex, these properties will allow us to deduce everything that we will need.

3.7 Example. Since étale morphisms are smooth with trivial Kähler differentials, if f is étale, then by (3.5.4) we have $\mathcal{L}_f \simeq 0$.

3.8 Example. Similarly, since closed immersions have trivial Kähler differentials, if i is a closed immersion, then by (3.5.4) we have $\mathrm{H}_0(\mathcal{L}_i) = 0$. Thus (3.5.5) shows that $\tau_{\leq 1}\mathcal{L}_i \simeq \mathcal{N}_i[1]$.

3.9 Remarks.

(3.9.1) The Tor-independence condition in (3.5.2) is really just saying that the ordinary pullback $Y' = X' \times_X Y$ of schemes is already derived.

(3.9.2) While (3.5.1)–(3.5.4) are automatic or easy from definitions of the cotangent complex, (3.5.5) requires work. Two approaches are to either relate the cotangent complex to square-zero extensions [8, Chapitre III, Corollaires 1.2.8.1 & 3.2.7], or use Quillen's *fundamental spectral sequence* [15, Theorem 6.3].

3.10 Remark. For a morphism of schemes $f\colon Y \to X$, the quasicoherent sheaves $\mathrm{H}_*(\mathcal{L}_f)$ on Y are known as the ***André–Quillen homology*** of f.

3.11 Remark (formally smooth and étale morphisms via the cotangent complex)**.** Let f be a morphism of schemes. Consider the following statements:

(3.11.1) The morphism f is smooth.

(3.11.2) The natural morphism $\mathcal{L}_f \to \Omega_f[0]$ is an equivalence and Ω_f is finite locally free. Equivalently, $\mathcal{L}_f$ is 0-truncated and perfect.

(3.11.3) The morphism f is formally smooth.

Then we have implications (3.11.1)⇒(3.11.2)⇒(3.11.3). Hence if f is locally of finite presentation, then the statements (3.11.1)–(3.11.3) are equivalent. However, the implication (3.11.3)⇒(3.11.2) is false in general: [STK, Tag 06E5] provides an example of a formally étale morphism with $\mathrm{H}_2(\mathcal{L}_f) \neq 0$.

Similarly, consider the following statements:

(3.11.4) The morphism f is étale.

(3.11.5) The cotangent complex $\mathcal{L}_f$ is trivial.

(3.11.6) The morphism f is formally étale.

Then we have implications (3.11.4)⇒(3.11.5)⇒(3.11.6). Hence if f is locally of finite presentation, then the statements (3.11.4)–(3.11.6) are equivalent. Again, the implication (3.11.6)⇒(3.11.5) is false.

See [8, Chapitre III, Propositions 3.1.1 & 3.1.2] for the details of the proofs of these implications.

4 Local complete intersections and the cotangent complex

In this section we show that the cotangent complex of a local complete intersection morphism is perfect and 1-truncated (Corollary 4.7). Through our proof, we show that if a morphism f admits a factorization as a composite $f = si$ where i is a regular immersion and s is smooth, then the cotangent complex of f is simply given by the complex $\mathcal{N}_i \to i^\star\Omega_s$ with differential coming from the conormal sequence (Proposition 4.3). We then state Avramov's characterization of lci morphisms in the locally noetherian setting in terms of the cotangent complex (Theorem 4.11).

We begin by showing that the cotangent complex of a local complete intersection is 1-truncated.

4.1 Proposition. *If $f\colon Y \to X$ is lci, then the cotangent complex $\mathcal{L}_f$ of f is 1-truncated.*

Proof sketch. Since the claim is local on Y, we can assume that X is affine and f factors as

$$\begin{tikzcd} Y \arrow[r, hook, "i"] \arrow[dr, "f"'] & V \arrow[d, "s"] \\ & X, \end{tikzcd}$$

where i is a regular closed immersion and s is smooth. Since $\mathcal{L}_s \simeq \Omega_s[0]$, the fiber sequence

$$i^\star\mathcal{L}_s \to \mathcal{L}_f \to \mathcal{L}_i$$

induces isomorphisms $\mathrm{H}_k(\mathcal{L}_f) \cong \mathrm{H}_k(\mathcal{L}_i)$ for all $k > 1$. So we can reduce to the case where $V = X = \mathrm{Spec}(A)$, and $Y = \mathrm{Spec}(A/(a_1, \dots, a_n))$, where $(a_1, \dots, a_n) \subset A$ is Koszul-regular. The fundamental fiber sequence allows us to inductively reduce to the case that $n = 1$.

That is, we have reduced to the case where $Y = \mathrm{Spec}(A/(a))$, where $a \in A$ is not a zerodivisor. The square

(4.2)
$$\begin{tikzcd} \mathrm{Spec}(A/(a)) \arrow[r] \arrow[d, hook] & \mathrm{Spec}(\mathbf{Z}[t]/(t)) \arrow[d, hook, "i"] \\ \mathrm{Spec}(A) \arrow[r] & \mathbf{A}^1_{\mathbf{Z}}, \end{tikzcd}$$

induced by the map $\mathbf{Z}[t] \to A$ sending $t \mapsto a$ is a pullback. Moreover, since a is not a zerodivisor, the square (4.2) is Tor-independent, so we can reduce to proving the claim for the cotangent complex of the zero section $i\colon \operatorname{Spec}(\mathbf{Z}) \hookrightarrow \mathbf{A}^1_{\mathbf{Z}}$. The fundamental fiber sequence applied to

$$\begin{tikzcd} \operatorname{Spec}(\mathbf{Z}) \arrow[r, hook, "i"] \arrow[dr, equal] & \mathbf{A}^1_{\mathbf{Z}} \arrow[d] \\ & \operatorname{Spec}(\mathbf{Z}) \end{tikzcd}$$

gives a fiber sequence

$$i^\star \mathcal{L}_{\mathbf{A}^1_{\mathbf{Z}}/\mathbf{Z}} \longrightarrow \mathcal{L}_{\operatorname{Spec}(\mathbf{Z})/\mathbf{Z}} \longrightarrow \mathcal{L}_i \,.$$

Since $\mathbf{A}^1_{\mathbf{Z}}$ is smooth over $\operatorname{Spec}(\mathbf{Z})$, we have $\mathcal{L}_{\mathbf{A}^1_{\mathbf{Z}}/\mathbf{Z}} \simeq \Omega_{\mathbf{A}^1_{\mathbf{Z}}/\mathbf{Z}}[0]$. The fact that $\mathcal{L}_{\mathbf{Z}/\mathbf{Z}} \simeq 0$ now implies that $\mathrm{H}_k(\mathcal{L}_i) = 0$ for $k > 1$, as desired. □

The next result gives a description of the 1-truncation of the cotangent complex of a morphism that admits a factorization into a closed immersion followed by a smooth morphism.

4.3 Proposition ([8, Chapitre III, Corollaire 3.2.7]). *Let $f\colon Y \to X$ be a morphism of schemes, and assume that f admits a factorization as $f = si$, where i is a closed immersion and s is smooth. Then there is a natural equivalence*

$$\tau_{\leq 1}\mathcal{L}_f \simeq [\, \mathcal{N}_i \xrightarrow{\mathrm{d}} i^\star \Omega_s \,] \,,$$

where $i^\star \Omega_s$ is in degree 0.

Proof. Since s is smooth, $\mathcal{L}_f \simeq \Omega_s[0]$, so the fundamental fiber sequence for the factorization $f = si$ is given by

$$i^\star \Omega_s[0] \longrightarrow \mathcal{L}_f \longrightarrow \mathcal{L}_i \,. \tag{4.4}$$

Since i is a closed immersion, $\mathrm{H}_0(\mathcal{L}_i) = 0$ and $\mathrm{H}_1(\mathcal{L}_i) \cong \mathcal{N}_i$, so $\tau_{\leq 1}\mathcal{L}_i \simeq \mathcal{N}_i[1]$. Applying $\tau_{\leq 1}$ to the fiber sequence we obtain a fiber sequence

$$i^\star \Omega_s[0] \longrightarrow \tau_{\leq 1}\mathcal{L}_f \longrightarrow \mathcal{N}_i[1] \,.$$

The connecting homomorphism $\mathcal{N}_i \to i^\star \Omega_s$ is given by $-\mathrm{d}\colon \mathcal{N}_i \to i^\star \Omega_s$, so we deduce that the cotangent complex $\mathcal{L}_f$ is given by the cofiber of $-\mathrm{d}\colon \mathcal{N}_i \to i^\star \Omega_s$ in $\mathbf{QCoh}(Y)$. Equivalently, $\mathcal{L}_f \simeq \operatorname{cofib}(\mathrm{d})$. To conclude, note that $\operatorname{cofib}(d)$ is simply given by the complex

$$\mathcal{N}_i \xrightarrow{\mathrm{d}} i^\star \Omega_s \,,$$

where $i^\star \Omega_s$ is in degree 0. □

4.5 Corollary. *Let $f: Y \to X$ be a morphism of schemes, and assume that f admits a factorization as $f = si$, where i is a regular closed immersion and s is smooth. Then there is a natural equivalence*

$$\mathcal{L}_f \simeq [\, \mathcal{N}_i \xrightarrow{\mathrm{d}} i^\star \Omega_s \,] \, .$$

Proof. Combine Propositions 4.1 and 4.3. □

4.6 Corollary. *Let $i: Z \hookrightarrow X$ be a regular closed immersion of schemes. Then $\mathcal{L}_i \simeq \mathcal{N}_i[1]$.*

4.7 Corollary. *If $f: Y \to X$ is lci, then the cotangent complex $\mathcal{L}_f$ is a 1-truncated perfect complex.*

Proof. All that remains is to show that $\mathcal{L}_f$ is perfect. The statement is local on Y, so we can assume that f factors through a regular closed embedding $i: Y \hookrightarrow V$ over X, where $s: V \to X$ is smooth. Corollary 4.5 provides an equivalence $\mathcal{L}_f \simeq [\mathcal{N}_i \to i^\star \Omega_s]$. Thus it suffices to shows that $\mathcal{N}_i$ and $i^\star \Omega_s$ are locally free of finite rank. The fact that s is smooth implies that Ω_s is locally free of finite rank [STK, Tag 00TH], and the fact that the ideal sheaf of i is regular implies that $\mathcal{N}_i$ is locally free of finite rank [STK, Tag 07CU]. □

4.8. Of particular use to us will be that the cotangent complex of an lci morphism $f: Y \to X$ defines a point of the K-theory ∞-groupoid $\mathrm{K}(Y)$.

Avramov's characterization of local complete intersections

Since smooth morphisms admit a characterization in terms of the cotangent complex (Remark 3.11), it is natural to ask if lci morphisms admit an analagous characterization.

4.9 Question. To what extent does Corollary 4.7 characterize lci morphisms in terms of the cotangent complex?

The following characterization of finite type lci morphisms with locally noetherian target is due to Lichtenbaum–Schlessinger and Quillen [8, Proposition 3.2.6; 9, Theorem 3.3.3; 15, Theorems 5.4 & 5.5].

4.10 Theorem. *Let $f: Y \to X$ be a morphism of schemes where X is locally noetherian. If f is locally of finite type, then f is lci if and only if the cotanget complex $\mathcal{L}_f$ is perfect and 1-truncated.*

If the source is also locally noetherian, the assumption that f be locally of finite type is unnecessary.

4.11 Theorem (Avramov [2, Theorem 1.2])**.** *Let $f: Y \to X$ be a morphism of locally noetherian schemes. Then f is lci if and only if the cotangent complex $\mathcal{L}_f$ is 1-truncated.*

4.12 Warning. The noetherianity assumptions in Theorem 4.11 are necessary. If k is a perfect field of characteristic $p > 0$, and X is a perfect k-scheme, then the cotangent complex $\mathcal{L}_{X/k}$ vanishes because the Frobenius needs to induce both an isomorphism on $\mathcal{L}_{X/k}$ and multiplication by p. So if $f: X \to Y$ is a morphism of perfect k-schemes, by the fundamental fiber sequence the cotangent complex $\mathcal{L}_f$ of f vanishes. However, not every morphism of perfect k-schemes is lci; the problem here is that perfect schemes are almost never (locally) noetherian.

We finish with one more relationship between lci morphisms and the cotangent complex. To do this, we need to introduce the *virtual relative dimension* of an lci morphism.

4.13 Definition ([SGA 6, Exposé VIII, Proposition 1.8 & Définition 1.9; 10, Chapter 6, Definition 4.11]). Let $f\colon Y \to X$ be an lci morphism and $y \in Y$. The ***virtual relative dimension of f at y*** is the rank

$$\mathrm{vrdim}_y(f) \coloneqq \mathrm{rank}_y\, \mathcal{L}_f\ .$$

Note that the virtual relative dimension is a locally constant integer-valued function on Y.

Let d be an integer. We say that f ***has virtual relative dimension*** d if the function $y \mapsto \mathrm{vrdim}_y(f)$ is constant with value d.

4.14 Remark. If we choose a factorization of f as a composite $f = si$ in a neighborhood of y where i is a regular immersion and s is smooth, then the virtual relative dimension at y is given by

$$\mathrm{vrdim}_y(f) = \mathrm{rank}_{i(y)}\, \Omega_s - \mathrm{rank}_y\, \mathcal{N}_i\ .$$

4.15 Proposition. *Let $f\colon Y \to X$ be an lci morphism of virtual relative dimension $d \geq 0$.*

(4.15.1) *The morphism f is syntomic if and only if the nonempty fibers of f have dimension d.*

(4.15.2) *If X and Y are affine, then f is a relative global complete intersection if and only if f is syntomic and $[\mathcal{L}_f] = [\mathcal{O}_Y^{\oplus d}]$ in $\mathrm{K}_0(Y)$.*

Proof. First we prove (4.15.1). Assume that f is flat. Since flat lci morphisms are stable under pullback, we can assume that X is the spectrum of a field. Then it is clear that Y has dimension d.

Now we prove the converse. Since flatness can be checked locally on X and Y, we can assume that X and Y are affine. Choose a closed immersion $i\colon Y \hookrightarrow \mathbf{A}^n_X$ over X. Since f is lci, by definition, i is regular, and hence the conormal sheaf $\mathcal{N}_i$ is locally free of rank $n-d$. Let $U \subset \mathbf{A}^n_X$ be an affine open for which $\mathcal{N}_i|_U$ is free. By Nakayama's lemma, there exists a function h on U such that $Y \times_{\mathbf{A}^n_X} U$ is cut out by $n-d$ equations in U_h. Lemma 2.7 shows that $Y \times_{\mathbf{A}^n_X} U \to X$ is a relative global complete intersection, hence flat (Lemma 2.4).

Now we prove (4.15.2). First assume that f is syntomic and $[\mathcal{L}_f] = [\mathcal{O}_Y^{\oplus d}]$ in $\mathrm{K}_0(Y)$. Since f is syntomic, we can choose a factorization of f as

(4.16)
$$\begin{tikzcd} Y \arrow[r, hook, "i"] \arrow[dr, "f"'] & \mathbf{A}^n_X \arrow[d] \\ & X\ , \end{tikzcd}$$

where i is a regular closed embedding. Corollary 4.5 shows that

$$[\mathcal{L}_f] = [\mathcal{O}_Y^{\oplus n}] - [\mathcal{N}_i]$$

in $\mathrm{K}_0(Y)$. Hence $[\mathcal{L}_f] = [\mathcal{O}_Y^{\oplus d}]$ if and only if the conormal sheaf $\mathcal{N}_i$ is stably free. In this case, by increasing n if necessary, we may assume that $\mathcal{N}_i$ is free, which shows that that f is a relative global complete intersection (Lemma 2.5).

Conversely, if f is a relative global complete intersection, then we can choose a factorization (4.16) such that the conormal sheaf $\mathcal{N}_i$ is free (Lemma 2.4). Corollary 4.5 then proves the claim. ☐

5 The lci locus of the Hilbert scheme of points

In this section we define the lci locus of the Hilbert functor of points, first introduced by Ciocan-Fontanine and Kapranov [4, §4.3]. Following [5, §5.1] we then prove that the lci locus of the Hilbert functor of points of a smooth S-scheme is formally smooth over S (Theorem 5.2). For another treatment of the lci locus of the Hilbert functor of points, see [STK, Tag 06CJ].

5.1 Definition. Let S be a scheme and $X \in \mathbf{Sch}_S$. The ***Hilbert functor of local complete intersections*** is the subfunctor

$$\mathrm{Hilb}^{\mathrm{flci}}(X/S) \subset \mathrm{Hilb}^{\mathrm{fin}}(X/S)$$

defined by sending $Y \in \mathbf{Sch}_S$ to the set of closed subschemes $Z \subset X \times_S Y$ that are finite syntomic over Y.

5.2 Theorem. *Let S be a scheme, $X \in \mathbf{Sch}_S$, and $d \geq 0$ an integer.*

(5.2.1) *The subfunctor* $\mathrm{Hilb}^{\mathrm{flci}}(X/S) \subset \mathrm{Hilb}^{\mathrm{fin}}(X/S)$ *is open.*

(5.2.2) *If $X \to S$ is smooth, then* $\mathrm{Hilb}^{\mathrm{flci}}(X/S)$ *is formally smooth over S.*

Proof. Assertion (5.2.1) is immediate from [7, Corollaire 19.3.8].

Now we prove (5.2.2). Let $V \subset V'$ be a first-order thickening of affine schemes and $V' \to S$ be a morphism. We need to show that every closed subscheme $i\colon Z \hookrightarrow V \times_S X$ that is finite syntomic over V can be lifted to a closed subscheme $Z' \subset V' \times_S X$ that is finite syntomic over V'. Since $X \to S$ is smooth, the immersion i is lci [STK, Tag 069M], and in particular the conormal sheaf $\mathcal{N}_i$ is finite locally free. Sicne Z is affine, the canonical sequence of conormal sheaves

$$0 \longrightarrow i^{\star}(\mathcal{N}_{V\times_S X/V'\times_S X}) \longrightarrow \mathcal{N}_{Z/V'\times_S X} \longrightarrow \mathcal{N}_i \longrightarrow 0$$

is split exact (see [STK, Tag 063N]). Choosing a splitting, we can identify $\mathcal{N}_i$ with a subsheaf of $\mathcal{N}_{Z/V'\times_S X}$. Let $Z^{(1)}$ be the first-order infintesimal neighborhood of Z in $V'\times_S X$, so that the conormal sheaf $\mathcal{N}_{Z/V'\times_S X}$ can be identified with the ideal sheaf $I_{Z\subset Z^{(1)}}$ of Z in $Z^{(1)}$.

Let $Z' \subset Z^{(1)}$ be the closed subscheme cut out by the submodule (hence subideal)

$$\mathcal{N}_i \subset \mathcal{N}_{Z/V'\times_S X} = I_{Z\subset Z^{(1)}} .$$

By construction, $\mathcal{N}_{Z/Z'}$ is the pullback of $\mathcal{N}_{V\times_S X/V'\times_S X}$ to Z, hence also the pullback to $\mathcal{N}_{V/V'}$ to Z. By [STK, Tag 06AG(23)], we further deduce that Z' is finite locally free over V', so Z' defines an element of $\mathrm{Hilb}^{\mathrm{fin}}(X/S)(V')$ lifting $Z \in \mathrm{Hilb}^{\mathrm{fin}}(X/S)(V)$. Since $V \to V'$ is surjective and open subfunctors are stable under generization, (5.2.1) implies that $Z' \to V'$ is lci. □

6 The Hilbert scheme of framed points

We now define *framings* and the Hilbert functor of *framed points*, which plays the role of the moduli space of framed 0-manifolds and cobordisms from topology. We finish by providing some examples.

6.1 Definition ([5, Definition 5.1.7])**.** Let X be a smooth S-scheme, and

$$[i\colon Z \hookrightarrow X\times_S T] \in \mathrm{Hilb}^{\mathrm{fin}}(X/S)(T)$$

A ***framing*** of Z is an isomorphism $\phi\colon \mathcal{N}_i \xrightarrow{\sim} i^\star(\Omega_{X\times_S T/T})$. The ***Hilbert scheme of framed points*** of X over S is the functor

$$\mathrm{Hilb}^{\mathrm{fr}}(X/S)\colon \mathbf{Sch}_S^{\mathrm{op}} \to \mathbf{Set}$$

sending $T \in \mathbf{Sch}_S$ to the set of pairs (Z,ϕ) with $Z \in \mathrm{Hilb}^{\mathrm{fin}}(X/S)(T)$ and ϕ a framing of Z.

6.2. Again, the degree induces a decomposition

$$\mathrm{Hilb}^{\mathrm{fr}}(X/S) \simeq \coprod_{d\geq 0} \mathrm{Hilb}^{\mathrm{fr}}_d(X/S)$$

of product-preserving presheaves on $\mathbf{Sch}_S$.

6.3. If $Z \in \mathrm{Hilb}^{\mathrm{fin}}(X/S)(T)$ admits a framing, then by Nakayama's lemma Z is locally cut out by a minimal number of equations, hence is a local complete intersection over T. Thus there is a map forgetting framings

$$\mathrm{Hilb}^{\mathrm{fr}}(X/S) \to \mathrm{Hilb}^{\mathrm{flci}}(X/S)\ .$$

6.4 Example. Let k be a field. Then we can identify $\mathrm{Hilb}^{\mathrm{fin}}_d(\mathbf{A}^1_k/k)(k)$ with the set of ideals $I \subset k[x]$ such that the quotient $k[x]/I$ is a k-vector space of dimension d. Since $k[x]$ is a principal ideal domain, every ideal I is generated by a polynomial f, so that

$$I/I^2 \cong (f)/(f^2)\ .$$

A framing of a point corresponding to an ideal $I = (f)$ is the data of an isomorphism

$$\begin{aligned}(f)/(f^2) &\xrightarrow{\sim} \Omega_{k[x]/k}\otimes_{k[x]}(k[x]/(f))\\ &\cong k[x]\,\mathrm{d}x/(f)\ .\end{aligned}$$

Multiplication by f provides such an isomorphism, so every k-point of $\mathrm{Hilb}^{\mathrm{fin}}_d(\mathbf{A}^1_k/k)(k)$ admits a framing (in particular, is lci).

In fact, $\mathrm{Hilb}^{\mathrm{flci}}(\mathbf{A}^1_k/k) = \mathrm{Hilb}^{\mathrm{fin}}(\mathbf{A}^1_k/k)$, but this is more difficult to see.

6.5 Warning. It is not true that $\mathrm{Hilb}^{\mathrm{flci}}(\mathbf{A}^n_k/k) = \mathrm{Hilb}^{\mathrm{fin}}(\mathbf{A}^n_k/k)$ for $n > 1$. For example, not every k-point of $\mathrm{Hilb}^{\mathrm{fin}}(\mathbf{A}^2_k/k)$ is lci. Consiser the quotient

$$k[x, y]/(x^2, xy, y^2) .$$

As a k-vector space $(x^2, xy, y^2)/(x^2, xy, y^2)^2$ has dimension 7, generated by x^2, xy, y^2, x^2y, xy^2, x^3, and y^3. On the other hand, since $\Omega_{k[x,y]/k}$ is a free $k[x, y]$-module of rank 2, for any ideal $I \subset k[x, y]$ such that $k[x, y]/I$ is a finite dimensional k-vector space, the k-vector space

$$\Omega_{k[x,y]/k} \otimes_{k[x,y]} (k[x, y]/I)$$

is even-dimensional.

References

STK The Stacks Project Authors, *Stacks project*, stacks.math.columbia.edu, 2019.

HA J. Lurie, *Higher algebra*, Preprint available at math.harvard.edu/~lurie/papers/HA.pdf, Sep. 2017.

SGA 6 *Théorie des intersections et théorème de Riemann-Roch*, Séminaire de Géométrie Algébrique du Bois Marie 1966–67 (SGA 6). Dirigé par P. Berthelot, A. Grothendieck, L. Illusie. Avec la collaboration de D. Ferrand, J. P. Jouanolou, O. Jussila, S. Kleiman, M. Raynaud, J. P. Serre. Lecture Notes in Mathematics, Vol. 225. Berlin: Springer-Verlag, 1966–67, pp. xii+700.

1. A. B. Altman and S. L. Kleiman, *Compactifying the Picard scheme*, Adv. in Math., vol. 35, no. 1, pp. 50–112, 1980. DOI: 10.1016/0001-8708(80)90043-2.
2. L. L. Avramov, *Locally complete intersection homomorphisms and a conjecture of Quillen on the vanishing of cotangent homology*, Ann. of Math. (2), vol. 150, no. 2, pp. 455–487, 1999. DOI: 10.2307/121087.
3. B. Bhatt, M. Morrow, and P. Scholze, *Topological Hochschild homology and integral p-adic Hodge theory*, Publ. Math. Inst. Hautes Études Sci., vol. 129, pp. 199–310, 2019. DOI: 10.1007/s10240-019-00106-9.
4. I. Ciocan-Fontanine and M. M. Kapranov, *Derived Hilbert schemes*, J. Amer. Math. Soc., vol. 15, no. 4, pp. 787–815, 2002. DOI: 10.1090/S0894-0347-02-00399-5.
5. E. Elmanto, M. Hoyois, A. A. Khan, V. Sosnilo, and M. Yakerson, *Motivic infinite loop spaces*, Preprint available at arXiv:1711.05248., Aug. 2019.
6. J. Fogarty, *Algebraic families on an algebraic surface*, Amer. J. Math, vol. 90, pp. 511–521, 1968. DOI: 10.2307/2373541.
7. A. Grothendieck, *Éléments de géométrie algébrique. IV. étude locale des schémas et des morphismes de schémas IV*, Inst. Hautes Études Sci. Publ. Math., no. 32, p. 361, 1967.
8. L. Illusie, *Complexe cotangent et déformations. I*, Lecture Notes in Mathematics, Vol. 239. Springer-Verlag, Berlin-New York, 1971, pp. xv+355.
9. S. Lichtenbaum and M. Schlessinger, *The cotangent complex of a morphism*, Trans. Amer. Math. Soc., vol. 128, pp. 41–70, 1967. DOI: 10.2307/1994516.

10. Q. Liu, *Algebraic geometry and arithmetic curves*, Oxford Graduate Texts in Mathematics. Oxford University Press, Oxford, 2002, vol. 6, pp. xvi+576, Translated from the French by Reinie Erné, Oxford Science Publications, ISBN: 0-19-850284-2.

11. M. Morrow, *Topological hochschild homology in arithmetic geometry*, Arizona Winter School notes available at swc.math.arizona.edu/aws/2019/2019MorrowNotes.pdf, 2019.

12. N. Nitsure, *Construction of Hilbert and Quot schemes*, in *Fundamental algebraic geometry*, Math. Surveys Monogr. Vol. 123, Amer. Math. Soc., Providence, RI, 2005, pp. 105–137.

13. M. Olsson and J. Starr, *Quot functors for Deligne–Mumford stacks*, in, 8, vol. 31, Special issue in honor of Steven L. Kleiman, 2003, pp. 4069–4096. DOI: 10.1081/AGB-120022454.

14. T. D. Papaïoannou, *Answer to math Overflow Question 24678: Why "syntomic" if "flat, locally of finite presentation, and local complete intersection" is already available?* MO: 24678, May 2010.

15. D. Quillen, *On the (co-) homology of commutative rings*, in *Applications of Categorical Algebra (Proc. Sympos. Pure Math., Vol. XVII, New York, 1968)*, Amer. Math. Soc., Providence, R.I., 1970, pp. 65–87.

16. D. Rydh, *Representability of Hilbert schemes and Hilbert stacks of points*, Comm. Algebra, vol. 39, no. 7, pp. 2632–2646, 2011. DOI: 10.1080/00927872.2010.488678.