POINCARÉ/KOSZUL DUALITY

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Contents

1. Outline of Proof

We would like to prove the following theorem

Theorem 1.1 (Ayala-Francis). *Let M_* be a zero-pointed n-manifold with M connected. Let $\mathcal{V}$ be a stable, presentable, symmetric monoidal ∞-category. Let A be an augmented n-disk algebra valued in $\mathcal{V}$. Then the Poincaré/Koszul duality arrow induces an equivalence of towers*

$$P_\bullet \int_{M_*} A \to \tau^{\leq k} \int^{M_*^\neg} \mathsf{Bar}^{(n)} A$$

If $\mathcal{V}$ has a t-structure, then

$$\int_{M_*} A \to \int^{M_*^\neg} \mathsf{Bar}^{(n)} A$$

is an equivalence.

We outline the proof of the theorem, including the steps we have already completed.

Step 1. (Andy's talk) Construct the Koszul dual $\mathsf{Bar}^{(n)}A$ of an augmented n-disk algebra and the Poincaré/Koszul duality map

$$\int_{M_*} A \to \int^{M_*^\neg} \mathsf{Bar}^{(n)} A$$

Step 2. (Dexter's talk) Define the cofiltrations $P_\bullet$. Compute its layers and figure out when it converges.

Step 3. (My last talk) Define the cofiltrations $\tau^{\leq \bullet}$. Compute its layers and figure out when it converges.

Step 4. (This talk) Check that there is an equivalence between the layers of $P_\bullet$ and $\tau^{\leq \bullet}$. Show that the Poincaré/Koszul duality map factors through a map of towers

$$P_\bullet \int_{M_*} A \to \tau^{\leq k} \int^{M_*^\neg} \mathsf{Bar}^{(n)} A$$

Deduce that the Poincaré/Koszul duality map induces an equivalence on towers and conclude that the Poincaré/Koszul duality map factors as follows

$$\begin{array}{ccc} \int_{M_*} A & \longrightarrow & \int^{M_*^\neg} \mathsf{Bar}^{(n)} \\ \downarrow & \overset{\simeq}{\nearrow} & \\ P_\infty \int_{M_*} A & & \end{array}$$

Hence the Poincaré/Koszul duality map is an equivalence exactly when $P_\bullet$ converges.

1.0.1. *Conventions.* Throughout we let M_* be a zero-pointed n-manifold, A and augmented n-disk algebra, C an augmented n-disk coalgebra, and k a finite cardinality. Everything is over a stable, $\otimes$-presentable, $\otimes$-cocomplete, $\otimes$-sifted, symmetric monoidal ∞-category $\mathcal{V}$.

2. Duality of Layers

Recall that for every k, we have fiber sequences

$$\mathsf{Conf}_k^{\mathrm{fr}}(M_*) \bigotimes_{\Sigma_k \wr O(n)} (LA)^{\otimes k} \to P_k \int_{M_*} A \to P_{k-1} \int_{M_*} A$$

and

$$\mathsf{Maps}^{\Sigma_k \wr O(n)} \left(\mathsf{Conf}_k^{\neg,\mathrm{fr}}(M_*), \left(\mathsf{cKer}^{\mathrm{aug}} C(\mathbb{R}^n_+)\right)^{\otimes k} \right) \to \tau^{\leq k} \int^{M_*^\neg} C \to \tau^{\leq k-1} \int^{M_*^\neg} C$$

We would like to show that these fiber sequences are equivalent in the case $C = \mathsf{Bar}^{(n)} A$. Toward this end, we prove the following which is a special case of Corollary 2.7.2 of [2].

Proposition 2.1. *There is an equivalence of functors* $\mathsf{Alg}_n^{\mathrm{aug}}(\mathcal{V}) \to \mathcal{V}$

$$\mathsf{Conf}_k^{\mathrm{fr}}(M_*) \bigotimes_{\Sigma_k \wr O(n)} (LA)^{\otimes k} \to \mathsf{Maps}^{\Sigma_k \wr O(n)} \left(\mathsf{Conf}_k^{\neg,\mathrm{fr}}(M_*), \left(\mathsf{cKer}^{\mathrm{aug}} (\mathsf{Bar}^{(n)} A)(\mathbb{R}^n_+) \right)^{\otimes k} \right)$$

Step A. Use Linear Poincaré Duality to reduce to showing

$$(\mathbb{R}^n)^+ \otimes LA \simeq \mathsf{cKer}^{\mathrm{aug}} \left((\mathsf{Bar}^{(n)} A)(\mathbb{R}^n_+) \right) \tag{1}$$

This is Corollary 2.29 of [6] and Theorem 2.7.1 of [2].

Step B. Check (1) in the case of a free algebra $A = \mathbb{F}_n V$. This calculation will involve showing that the Koszul dual of a free thing is trivial.

Step C. Deduce the general case of (1) from the free case. The hard part here will be showing that $\mathsf{Bar}^{(n)}$ commutes with geometric realization.

Recall the statement of Linear Poincaré Duality, a proof of which can be found in Corollary 3.6.2 of [3].

Theorem 2.2 (Linear Poincaré Duality)**.** *Let E and F be $O(n)$-modules in $\mathcal{V}$. Let $\alpha : (\mathbb{R}^n)^+ \otimes E \to F$ be an equivalence of $O(n)$-modules, were $(\mathbb{R}^n)^+ \otimes E$ is given the diagonal $O(n)$-module structure. Then α induces an equivalence*

$$\mathsf{Fr}_{M_*} \bigotimes_{O(n)} E \to \mathsf{Maps}^{O(n)}(\mathsf{Fr}_{M_*^\neg}, F)$$

If W_* is a $B(\Sigma_k \wr O(n))$-structured zero-pointed manifold, then there is an equivalence

$$\mathsf{Fr}_{W_*} \bigotimes_{\Sigma_k \wr O(n)} E^{\otimes k} \to \mathsf{Maps}^{\Sigma_k \wr O(n)}(\mathsf{Fr}_{W_*^\neg}, F^{\otimes k})$$

Here Fr_{M_*} is the frame bundle of the zero-pointed manifold M_*. See Definition 1.6.1 of [2], or Andy's talk notes.

We would like to apply Theorem 2.2 to $W_* = \mathsf{Conf}_k^{\mathrm{fr}}(M_*)$. As noted in the previous talk, we can fix the problem that $\mathsf{Conf}_k(M_*)$ is not a zero-pointed manifold by replacing it be a Σ_k-homotopy equivalent space $C_k(M_*)$. See Lemma 1.2.3 of [2]. Applying Linear Poincaré Duality to $W = *C_k(M_*)$, and using the Σ_k-homotopy equivalences

$$C_k(M_*) \simeq \mathsf{Conf}_k(M_*)$$

and

$$C_k(M_*)^{\neg} \simeq \mathsf{Conf}_k^{\neg}(M_*)$$

we obtain and equivalence

$$\mathsf{Conf}_k^{\mathrm{fr}}(M_*) \bigotimes_{\Sigma_k \wr O(n)} E^{\otimes k} \to \mathsf{Maps}^{\Sigma_k \wr O(n)}\left(\mathsf{Conf}_k^{\neg,\mathrm{fr}}(M_*), F^{\otimes k}\right)$$

where $\mathsf{Conf}_k^{\mathrm{fr}}(M_*) := \mathsf{Fr}_{\mathsf{Conf}_k(M_*)}$ and $\mathsf{Conf}_k^{\neg,\mathrm{fr}}(M_*) := \mathsf{Fr}_{\mathsf{Conf}_k^{\neg}(M_*)}$.

This completes Step A of the proof of Proposition 2.1.

2.1. **Free Case.**

We have reduced proving Proposition 2.1 to proving the following

Theorem 2.3 (Francis)**.** *There is an equivalence of functors* $\mathsf{Alg}_n^{\mathrm{aug}}(\mathcal{V}) \to \mathsf{Mod}_{O(n)}(\mathcal{V})$

$$(\mathbb{R}^n)^+ \otimes L(-) \xrightarrow{\sim} \mathsf{cKer}^{\mathrm{aug}}(\mathsf{Bar}^{(n)}(-)(\mathbb{R}^n_+))$$

The point of this section is to complete Step B of the proof of Proposition 2.1: to prove Francis' theorem in the free case. Note that the cotangent functor L is the left adjoint to the trivial augmented algebra functor and evaluation on $\mathbb{R}^n_+$ is the right adjoint to the free functor. Since the composition

$$\mathsf{Mod}_{O(n)}(\mathcal{V}_{\mathbb{1}//\mathbb{1}}) \xrightarrow{t^{\mathrm{aug}}} \mathsf{Alg}_n^{\mathrm{aug}}(\mathcal{V}) \xrightarrow{\mathsf{ev}_{\mathbb{R}^n_+}} \mathsf{Mod}_{O(n)}(\mathcal{V}_{\mathbb{1}//\mathbb{1}})$$

is the identity, the same holds for the composition of the adjoints,

$$\mathsf{Mod}_{O(n)}(\mathcal{V}_{\mathbb{1}//\mathbb{1}}) \xrightarrow{\mathbb{F}_n^{\mathrm{aug}}} \mathsf{Alg}_n^{\mathrm{aug}}(\mathcal{V}) \xrightarrow{L^{\mathrm{aug}}} \mathsf{Mod}_{O(n)}(\mathcal{V}_{\mathbb{1}//\mathbb{1}})$$

Thus

$$L\mathbb{F}_n V = L\mathbb{F}_n^{\mathrm{aug}}(\mathbb{1} \oplus V) \simeq \mathsf{ker}^{\mathrm{aug}} L^{\mathrm{aug}}\mathbb{F}_n^{\mathrm{aug}}(\mathbb{1} \oplus V) = \mathsf{ker}^{\mathrm{aug}}(\mathbb{1} \oplus V) = V$$

Therefore, in the free case we can restate Theorem 2.3 as an equivalence

$$\mathbb{1} \oplus \left((\mathbb{R}^n)^+ \otimes V\right) \simeq \left(\mathsf{Bar}^{(n)}(\mathbb{F}_n V)\right)(\mathbb{R}^n_+)$$

So, if Theorem 2.3 is true, then the symmetric monoidal functor $\mathsf{Bar}^{(n)}(\mathbb{F}_n V) \colon \mathsf{Disk}_n^+ \to \mathcal{V}$ must agree on objects with the trivial augmented coalgebra $t^{\mathrm{aug}}_{\mathrm{cAlg}}((\mathbb{R}^n)^+ \otimes V)$. In fact, more is true: these functors agree on morphisms as well. This is a general feature of Koszul duality; the Koszul dual of a free algebra is a trivial coalgebra.

Lemma 2.4. *Let V be an $O(n)$-module in $\mathcal{V}$. There is an equivalence of augmented n-disk coalgebras*

$$\mathsf{Bar}^{(n)}(\mathbb{F}_n V) \simeq t^{\mathrm{aug}}_{\mathrm{cAlg}}((\mathbb{R}^n)^+ \otimes V)$$

This is Lemma 2.4.3 of [2]. We will only show that the underlying $O(n)$-modules in $\mathcal{V}$ agree, since that is all we will use today.

Proof. We check that the two symmetric monoidal functors $\mathsf{Disk}_n^+ \to \mathcal{V}$ agree on objects. Since they are symmetric monoidal, it suffices to check that they agree on $(\mathbb{R}^n)^+$. By definition (or by Theorem 3.3.2 of [3]), we have

$$\mathsf{Bar}^{(n)}(\mathbb{F}_n V)((\mathbb{R}^n)^+) = \int_{(\mathbb{R}^n)^+} \mathbb{F}_n V$$

By Theorem 2.4.1 of [2], we have

$$\int_{(\mathbb{R}^n)^+} \mathbb{F}_n V = \bigoplus_{i\geq 0} \mathsf{Conf}_i^{\mathrm{fr}}((\mathbb{R}^n)^+) \bigotimes_{\Sigma_i \wr O(n)} (L\mathbb{F}_n V)^{\otimes i} = \bigoplus_{i\geq 0} \mathsf{Conf}_i^{\mathrm{fr}}((\mathbb{R}^n)^+) \bigotimes_{\Sigma_i \wr O(n)} V^{\otimes i}$$

Claim 1. *For every $i > 1$, the configuration space $\mathsf{Conf}_i((\mathbb{R}^n)^+)$ is contractible.*

Assuming this for a moment, we obtain an equivalence

$$\int_{(\mathbb{R}^n)^+} \mathbb{F}_n V \simeq \mathbb{1} \oplus \mathsf{Conf}_1^{\mathrm{fr}}((\mathbb{R}^n)^+) \bigotimes_{O(n)} V$$

Now $\mathsf{Conf}_1((\mathbb{R}^n)^+) = (\mathbb{R}^n)^+$. Hence

$$\mathsf{Bar}^{(n)}(\mathbb{F}_n V)((\mathbb{R}^n)^+) \simeq \mathbb{1} \oplus (\mathbb{R}^n)^+ \otimes V$$

□

For a proof, see Proposition 6.1.12 of [7] or Lemma 120 of [8].

This completes Step B of the proof of Proposition 2.1.

2.2. **General Case.** We deduce the general case of Theorem 2.3 from the free case.

The following is Lemma 2.5.2 of [2].

Lemma 2.5 (Free Resolutions)**.** *Every augmented n-disk algebra in $\mathcal{V}$ is a sifted colimit of free augmented n-disk algebras.*

We will write $A \simeq |\mathbb{F}^{\bullet+1}A|$ for the canonical choice of such a resolution.

Theorem 2.3 for a general augmented n-disk algebra A will follow from the case when A is free if the functors involved all preserve geometric realizations. Since the cotangent space functor L is a left adjoint, it preserves colimits and hence

$$LA = L(|\mathbb{F}^{\bullet+1}A|) = |L\mathbb{F}^{\bullet+1}A|$$

The main result of this section is the following,

Lemma 2.6. *The n-fold bar construction commutes with geometric realization*

$$\mathsf{Bar}^{(n)}(|A^\bullet|) = |\mathsf{Bar}^{(n)}(A^\bullet)|$$

This follows from the fact that Δ^{op} is sifted. See also Dexter's talk. Assuming this, we can prove Theorem 2.3.

Proof of Theorem 2.3. Let A be an augmented n-disk algebra. Then

$$\begin{aligned}
(\mathbb{R}^n)^+ \otimes LA &= (\mathbb{R}^n)^+ \otimes L(|\mathbb{F}_n^{\bullet+1}A|) \\
&= (\mathbb{R}^n)^+ \otimes |L\mathbb{F}_n^{\bullet+1}A| \\
&= |(\mathbb{R}^n)^+ \otimes L\mathbb{F}_n^{\bullet+1}A| \\
&\simeq |\mathsf{cKer}^{\mathrm{aug}}(\mathsf{Bar}^{(n)}(\mathbb{F}_n^{\bullet+1}A)(\mathbb{R}^n_+))| \\
&\simeq \mathsf{cKer}^{\mathrm{aug}}(\mathsf{Bar}^{(n)}(|\mathbb{F}_n^{\bullet+1}A|)(\mathbb{R}^n_+)) \\
&\simeq \mathsf{cKer}^{\mathrm{aug}}(\mathsf{Bar}^{(n)}A(\mathbb{R}^n_+))
\end{aligned}$$

$\square$

This completes Step C of the proof, and hence the whole proof, of Proposition 2.1.

3. Proof of the Main Theorem

We complete Step 4 of the proof of Theorem 1.1, and hence the proof of the Poincaré/Koszul duality theorem.

Lemma 3.1. *The Poincaré/Koszul duality map factors through the Goodwillie and cardinality cofiltrations,*

$$\begin{array}{ccc}
\int_{M_*} A & \longrightarrow & \int^{M_*^\neg} \mathsf{Bar}^{(n)} A \\
\downarrow & & \downarrow \\
\vdots & & \vdots \\
\downarrow & & \downarrow \\
P_k \int_{M_*} A & \dashrightarrow & \tau^{\leq k} \int^{M_*^\neg} \mathsf{Bar}^{(n)} A \\
\downarrow & & \downarrow \\
P_{k-1} \int_{M_*} A & \dashrightarrow & \tau^{\leq k-1} \int^{M_*^\neg} \mathsf{Bar}^{(n)} A
\end{array}$$

Proof. By the universal property of the Goodwillie kth Taylor approximation $P_k \int_{M_*}(-)$, it suffices to show that $\tau^{\leq k} \int^{M_*^\neg} \mathsf{Bar}^{(n)}(-)$ is polynomial of degree k. Using induction, it suffices to show that the layers

$$\mathsf{Maps}^{\Sigma_k \wr O(n)} \left(\mathsf{Conf}_k^{\neg,\mathrm{fr}}(M_*), \left(\mathsf{cKer}^{\mathrm{aug}} \mathsf{Bar}^{(n)} A(\mathbb{R}_+^n) \right)^{\otimes k} \right) \to \tau^{\leq k} \int^{M_*^\neg} \mathsf{Bar}^{(n)} A \to \tau^{\leq k-1} \int^{M_*^\neg} \mathsf{Bar}^{(n)} A$$

are homogeneous of degree k. By Proposition 2.1, we can identify the kth layer as

$$\mathsf{Conf}_k^{\mathrm{fr}}(M_*) \bigotimes_{\Sigma_k \wr O(n)} (LA)^{\otimes k} \to \mathsf{Maps}^{\Sigma_k \wr O(n)} \left(\mathsf{Conf}_k^{\neg,\mathrm{fr}}(M_*), \left(\mathsf{cKer}^{\mathrm{aug}} (\mathsf{Bar}^{(n)} A)(\mathbb{R}_+^n) \right)^{\otimes k} \right)$$

The left-hand side above is the kth layer of the Goodwillie tower, and hence is homogeneous of degree k. $\square$

Corollary 3.2. *The Poincaré/Koszul duality map is an equivalence on towers,*

$$P_\bullet \int_{M_*} A \to \tau^{\leq k} \int^{M_*^\neg} \mathsf{Bar}^{(n)} A$$

Proof. Proposition 2.1 says that the Poincaré/Koszul duality map induces an equivalence on the layers of the Goodwillie and cardinality towers. $\square$

Corollary 3.3 (Poincaré/Koszul Duality)**.** *If $\mathcal{V}$ has a t-structure, then*

$$\int_{M_*} A \to \int^{M_*^\neg} \mathsf{Bar}^{(n)} A$$

is an equivalence.

Proof. By Lemma 3.1 and Corollary 3.2, the Poincaré/Koszul duality map factors through an equivalence,

$$\begin{array}{ccc} \int_{M_*} A & \longrightarrow & \int^{M_*^{\neg}} \mathsf{Bar}^{(n)} A \\ \downarrow & & \downarrow \\ P_\infty \int_{M_*} A & \xrightarrow[\sim]{} & \tau^{\leq \infty} \int^{M_*^{\neg}} \mathsf{Bar}^{(n)} A \end{array}$$

Since the cardinality filtration always converges, the right vertical arrow is an equivalence. The left vertical arrow is an equivalence when $\mathcal{V}$ has a t-structure, as shown in Dexter's talk. □

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