A CARDINALITY FILTRATION

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Contents

1. Preliminaries 1
1.1. Stratified Spaces 1
1.2. Exiting Disks 2
2. Recollections 2
2.1. Linear Poincaré Duality 3
3. The Ran Space 5
4. Disk Categories 6
4.1. Cardinality Filtration 7
5. Layer Computations 9
5.1. Reduced Extensions 13
6. Convergence 15
References 16

1. Preliminaries

1.1. **Stratified Spaces.** Two talks ago, we discussed zero-pointed manifolds M_*. For parts of what we will do today, we will need to consider a wider class of spaces: (conically smooth) stratified spaces.

Definition 1.1. Let P be a partially ordered set. Topologize P by declaring sets that are closed upwards to be open. A P-stratified space is a topological space X together with a continuous map $f : X \to P$. We refer to the fiber $f^{-1}\{p\} =: X_p$ as the p-stratum.

We view zero-pointed manifolds as stratified spaces over $\{0 < 1\}$ with $*$ in the 0 stratum and $M = M_* \setminus *$ in the 1 stratum.

Definition 1.2. A stratified space $X \to P$ is called *conically stratified* if for every $p \in A$ and $x \in X_p$ there exists an $P_{>}p$-stratified space U and a topological space Z together with an open embedding

$$C(Z) \times U \to X$$

of stratified spaces whose image contains x. Here $C(Z) = * \coprod_{Z\times\{0\}} (Z \times \mathbb{R}_{\geq 0})$.

We will not discuss the definition of conically *smooth* stratified space. For now, just think of conically smooth stratified spaces as conically stratified spaces that are nice and smooth like manifolds. Details can be found in [4].

1.2. **Exiting Disks.** As usual, everything is over a stable, presentable, symmetric monoidal ∞-category $\mathcal{V}$.

For a zero-pointed n-manifold M_*, we have defined factorization homology over M_* with coefficients in A to be the colimit over the over category

$$\mathsf{colim}\left(\mathcal{D}\mathsf{isk}_{n,+/M_*} \xrightarrow{A} \mathcal{V}\right)$$

For this talk, we will replace the over category by a category of "exiting disks." Since we have been pretty lax about all the stratified details, I'm only going to give a pretend definition. Details can be found in [4] and [5].

Definition 1.3. A *basic* is a stratified space of the form $C(Z) \times \mathbb{R}^i$ for i a finite cardinality, Z a compact stratified space, and $C(Z)$ the open cone

$$C(Z) = * \bigsqcup_{Z \times \{0\}} Z \times [0, 1)$$

The enter-path category of a stratified space X is defined to be the category of basics in X,

$$\mathsf{Entr}(X) \colon = \mathcal{B}\mathsf{sc}_{/X}$$

Definition 1.4 (Pretend Definition)**.** Let X be a conically smooth stratified space with a distinguished base point $* \in X$. Define an ∞-category $\mathcal{D}\mathsf{isk}_+(X)$ with objects conically smooth embeddings

$$C(Z) \sqcup U \hookrightarrow X$$

where U is a finite disjoint union of copies of $\mathbb{R}^i$, and $C(Z)$ is the open cone on some compact neighborhood Z of $* \in X$. Morphisms in $\mathcal{D}\mathsf{isk}_+(X)$ are isotopies over X.

Remark. If we take $X = M_*$ to be a zero-pointed n-manifold, then the condition that $C(Z) \sqcup U \hookrightarrow M_*$ be a conically smooth embedding implies that U is a finite disjoint union of copies of n-dimensional euclidean space.

In more complicated examples, X might have strata of varying dimensions.

Heuristically, think of objects $\mathcal{D}\mathsf{isk}_+(M_*)$ as the same as $\mathcal{D}\mathsf{isk}_{n,+/M_*}$ but where we made the "niceness" around the zero point $*$ more explicit.

The following is Theorem 1.4.4 of [2]

Theorem 1.5. *The category $\mathcal{D}\mathsf{isk}_+(M_*)$ is sifted and the functor*

$$\mathcal{D}\mathsf{isk}_+(M_*) \to (\mathcal{D}\mathsf{isk}_{n,+})_{/(M_*)}$$

sending $C(Z) \times U$ to U_+, is final.

Thus we can compute $\int_{M_*} A$ as a colimit over $\mathcal{D}\mathsf{isk}_+(M_*)$.

2. Recollections

The goal of this seminar is to show that the Koszul duality arrow

$$\int_{M_*} A \to \int^{M_*^\neg} \mathsf{Bar}^{(n)} A$$

is an equivalence in certain situations. Last time, we considered a cofiltration of the left hand side by its Goodwillie polynomial approximations. Why does anyone filter anything? In hopes that the pieces are easier to study. Today we look for a cofiltration to fit on the right hand side of the

Poincaré/Koszul duality statement. Next time we will show that Poincaré/Koszul duality can be refined to an equivalence of towers

$$P_\bullet \int_{M_*} A \xrightarrow{\sim} \tau^{\leq \bullet} \int^{M_*^\neg} \mathsf{Bar}^{(n)} A$$

where the $\tau^{\leq \bullet} \int^{M_*^\neg}$ cofiltration is the one we will define today.

If we want to stick to factorization *homology*, we can recast this as

$$\left(P_\bullet \int_{M_*} A \right)^\vee \simeq \tau^{\leq \bullet} \int_{M_*} \mathbb{D}^n A \tag{1}$$

In this setting, $\tau^{\leq \bullet} \int_{M_*}$ is a filtration. In this talk, we will try to stay on the homology side.

2.1. **Linear Poincaré Duality.** If we had a duality of cofiltrations as in (1), then we would have an equivalence in $\mathcal{V}$ between the layers of $P_\bullet \int_{M_*}$ and $\tau^{\leq \bullet} \int_{M_*}$. Recall from last time that for each $k \geq 0$ there is a fiber sequence

$$\mathsf{Conf}_k^{\mathrm{fr}}(M_*) \bigotimes_{\Sigma_k \wr O(n)} (LA)^{\otimes k} \to P_k \int_{M_*} A \to P_{k-1} \int_{M_*} A$$

So we would like a cofiber sequence

$$\tau^{\leq k-1} \int_{M_*} A \to \tau^{\leq k} A \to \left(\mathsf{Conf}_k^{\mathrm{fr}}(M_*) \bigotimes_{\Sigma_k \wr O(n)} (LA)^{\otimes k} \right)^\vee$$

Linear Poincaré duality (which we saw in Andy's talk[1]) tells us that in $\mathcal{V}$ we have an equivalence

$$\mathsf{Conf}_k^{\mathrm{fr}}(M_*) \bigotimes_{\Sigma_k \wr O(n)} (LA)^{\otimes k} \xrightarrow{\sim} \mathsf{Maps}^{\Sigma_k \wr O(n)} \left(\mathsf{Conf}_k^{\mathrm{fr},\neg}(M_*), ((\mathbb{R}^n)^+ \otimes LA)^{\otimes k} \right)$$

We can rewrite this as

$$\left(\mathsf{Conf}_k^{\mathrm{fr}}(M_*) \bigotimes_{\Sigma_k \wr O(n)} (LA)^{\otimes k} \right)^\vee \simeq \mathsf{Conf}_k^{\mathrm{fr},\neg}(M_*) \bigotimes_{\Sigma_k \wr O(n)} \left((\mathbb{R}^n)^+ \otimes LA \right)^{\otimes k}$$

Let's ignore the A situation for a second by focusing on A so that $LA = \mathbb{1}_{\mathcal{V}}$. To simplify things further, take $M_* = M^+$ so that $\mathsf{Conf}_k^\neg(M_*) = \mathsf{Conf}_k(M)^+$. In this case, we're looking for a filtration with layers

$$\left(\mathsf{Conf}_k(M)^+_{\Sigma_k} \right) [nk]$$

which for $\mathcal{V} = \mathsf{Ch}$ is chains on the one-point compactification of the unordered configuration space, with a shift.

When LA is something interesting, we use the following theorem which is stated in Theorem 2.7.1 of [2] and proved in Corollary 2.29 of [6].

Theorem 2.1 (Francis)**.**

$$\mathbb{1} \oplus LA[n] \simeq (\mathsf{Bar}^{(n)} A)(\mathbb{R}^n)$$

This will be discussed next week.

Back to the case when we can ignore A. What are these layers filtering?

Warning. The next page or so has an overload of detail since this really confused me for a long time. Hopefully it's all correct here, but who knows.

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[1]We added in an action of the symmetric group Σ_k from the version stated in Andy's talk. See Remark 1.6.4 of [2].

View $\mathbb{1}_\mathcal{V}$ as an n-disk algebra in $\mathcal{V}$ by

$$\mathbb{1}_\mathcal{V}(\mathbb{R}^n) = \mathbb{1}_\mathcal{V}$$

and sending morphisms to the identifications $\mathbb{1}_\mathcal{V}^{\otimes k} \simeq \mathbb{1}_\mathcal{V}$.

Lemma 2.2. *We have an equivalence*

$$\int_M \mathbb{1}_\mathcal{V} \simeq B\mathcal{D}\mathsf{isk}_{n/M}$$

Proof. The left hand side can be computed as the homotopy colimit over the ordinary category $\mathsf{Disk}_{n/M}$,

$$\int_M \mathbb{1}_\mathcal{V} = \mathsf{hocolim}\left(\mathsf{Disk}_{n/M} \to \mathsf{Disk}_n \xrightarrow{\mathbb{1}_\mathcal{V}} \mathcal{V}\right)$$

The homotopy colimit of the unit is the classifying space. See Appendix B of [7] for a nice discussion of this. □

In the zero-pointed case, we instead consider the augmented n-disk algebra $\mathbb{1}_\mathcal{V} \times \mathbb{1}_\mathcal{V}$ given on $\mathbb{R}^n_+$ by

$$(\mathbb{1}_\mathcal{V} \times \mathbb{1}_\mathcal{V})(\mathbb{R}^n_+) = \mathbb{1}_\mathcal{V} \times \mathbb{1}_\mathcal{V}$$

On morphisms $\bigvee_k \mathbb{R}^n_+ \to \mathbb{R}^n_+$, define

$$\begin{array}{ccc} (\mathbb{1}_\mathcal{V} \times \mathbb{1}_\mathcal{V})(\bigvee_k \mathbb{R}^n_+) & \longrightarrow & (\mathbb{1}_\mathcal{V} \times \mathbb{1}_\mathcal{V})(\mathbb{R}^n_+) \\ \| & & \| \\ (\mathbb{1}_\mathcal{V} \times \mathbb{1}_\mathcal{V})^{\otimes k} & \longrightarrow & \mathbb{1}_\mathcal{V} \times \mathbb{1}_\mathcal{V} \end{array}$$

sending $((a_1, b_1), \ldots, (a_k, b_k))$ to $(a_1 \cdots a_k, b_1 \cdots b_k)$. The augmented n-disk algebra $\mathbb{1}_\mathcal{V} \times \mathbb{1}_\mathcal{V}$ is augmented by projection onto the left factor $\mathbb{1}_\mathcal{V} \times \mathbb{1}_\mathcal{V} \to \mathbb{1}_\mathcal{V}$.

Remark. The notation $\mathbb{1}_\mathcal{V} \times \mathbb{1}_\mathcal{V}$ is used instead of the notation $\mathbb{1}_\mathcal{V} \oplus \mathbb{1}_\mathcal{V}$ since $\mathbb{1}_\mathcal{V} \oplus W$ is sometimes used to denote the trivial augmented n-disk algebra on W. The algebra $\mathbb{1}_\mathcal{V} \times \mathbb{1}_\mathcal{V}$ is not a trivial algebra.

Lemma 2.3. *We have an equivalence*

$$\int_{M_*} \mathbb{1}_\mathcal{V} \times \mathbb{1}_\mathcal{V} = B\mathcal{D}\mathsf{isk}_{n,+/M_*}$$

Proof Idea. Take $M_* = M_+$. By Lemma 2.2.2 of [3] We have an equivalence

$$\int_M \mathbb{1}_\mathcal{V} \times \mathbb{1}_\mathcal{V} \xrightarrow{\sim} \int_{M_+} \mathbb{1}_\mathcal{V} \times \mathbb{1}_\mathcal{V}$$

where on the left-hand side we view $\mathbb{1}_\mathcal{V} \times \mathbb{1}_\mathcal{V}$ as a (non-augmented) n-disk algebra by

$$\mathcal{D}\mathsf{isk}_n \xrightarrow{(-)_+} \mathcal{D}\mathsf{isk}_{n,+} \xrightarrow{\mathbb{1}_\mathcal{V} \times \mathbb{1}_\mathcal{V}} \mathcal{V}$$

Now we have a commutative diagram

$$\begin{array}{ccc} \mathcal{D}\mathsf{isk}_n & \xrightarrow{\mathbb{1}_\mathcal{V} \times \mathbb{1}_\mathcal{V}} & \mathcal{V} \\ & \searrow^{\mathbb{1}_\mathcal{V}} & \uparrow{\scriptstyle \mathbb{1}_\mathcal{V} \times (-)} \\ & & \mathcal{V} \end{array}$$

Thus

$$\int_M \mathbb{1}_\mathcal{V} \times \mathbb{1}_\mathcal{V} = \mathbb{1}_\mathcal{V} \times \left(\int_M \mathbb{1}_\mathcal{V}\right) = \mathbb{1}_\mathcal{V} \times B\mathcal{D}\mathsf{isk}_{n/M} = B\mathcal{D}\mathsf{isk}_{n,+/M_+}$$

$\square$

Proposition 2.4. *The category $\mathcal{D}\mathsf{isk}_{n,+/M_*}$ is weakly contractible.*

Proof. By Proposition 5.5.8.7 of [**?**], every sifted ∞-category is weakly contractible. The category in question is sifted. By Corollary 3.22 of [1] or Proposition 5.5.2.15 of [9], the non-pointed disk category $\mathcal{D}\mathsf{isk}_{n/M}$ is sifted. For the pointed version, we have by Corollary 2.3.6 of [3], that the category $\mathcal{D}\mathsf{isk}_+(M_*)$ is sifted. By Theorem 1.5, there is a final functor

$$\mathcal{D}\mathsf{isk}_+(M_*) \to \mathcal{D}\mathsf{isk}_{n,+/M+*}$$

The diagonal functors for these two categories fit into a commutative diagram

$$\begin{array}{ccc} \mathcal{D}\mathsf{isk}_+(M_*) & \longrightarrow & \mathcal{D}\mathsf{isk}_{n,+/M_*} \\ \downarrow{\scriptstyle\Delta} & & \downarrow{\scriptstyle\Delta} \\ \mathcal{D}\mathsf{isk}_+(M_*) \times \mathcal{D}\mathsf{isk}_+(M_*) & \longrightarrow & \mathcal{D}\mathsf{isk}_{n,+/M_*} \times \mathcal{D}\mathsf{isk}_{n,+/M_*} \end{array}$$

The two horizontal functors are final by Theorem 1.5. The left vertical arrow is horizontal since $\mathcal{D}\mathsf{isk}_+(M_*)$ is sifted. By the two-out-of-three property for finality, the right vertical functor is final. Thus $\mathcal{D}\mathsf{isk}_{n,+/M_*}$ is sifted and hence weakly contractible. $\square$

Thus we are looking for a filtration of a point with layers that are the one point compactifications of configuration spaces. In other words, we are looking for a way to glue together configuration spaces into a contractible space.

3. The Ran Space

We introduce a contractible space $\mathsf{Ran}(M_*)$ with a natural filtration whose layers are $\mathsf{Conf}_k(M_*)^+$. Because we are missing/ignoring a bunch of background on stratified spaces, the topology in the following definition is only pretend.[2]

Definition 3.1 (Pretend Definition)**.** Let M_* be a zero-pointed manifold. Define a pointed space $\mathsf{Ran}(M_*)$ consisting of finite based subsets S_+ of M_*, with $+ \mapsto *$ and so that the map $S_+ \to \pi_0(M_*)$ is surjective. Give $\mathsf{Ran}(M_*)$ the coarsest topology for which the subspace

$$\{S_+ \in \mathsf{Ran}(M_*) : S_+ \subset \bigcup_{1\le i\le r} U_i \text{ and for each } i,\, S_+ \cap U_i \neq \emptyset\}$$

is open for any finite collection of disjoint open subspaces $\{U_i\}_{1\le i\le r}$ of M. The space $\mathsf{Ran}(M_*)$ is called the "Ran space" of M_*.

We consider a relative version of the Ran space of M_* containing a fixed subset of M_*. Let $T_+ \in \mathsf{Ran}(M_*)$ be a finite subset of M_* containing $*$. Let $\mathsf{Ran}(M_*)_{T_+}$ be the subset of $\mathsf{Ran}(M_*)$ consisting of pointed sets $S_+ \subset M_*$ that contain T_+.

Lemma 3.2 (Beilinson-Drinfeld)**.** *Let M connected and $T_+ \subset \mathsf{Ran}(M_*)$. The space $\mathsf{Ran}(M_*)_{T_+}$ is contractible.*

This is Lemma 5.5.1.8 of [9].

[2]The topology we define here is from Definition 5.5.1.2 of [9], which defines the Ran space of an ordinary manifold. Presumably these two topologies agree for a connected manifold, but I don't know of a direct reference for this.

Proof. Define an H-space structure on $\mathsf{Ran}(M_*)_{T_+}$ by union. Considering homotopy groups based at T_+, let ϕ be the induced map

$$\phi : \pi_n \mathsf{Ran}(M_*)_{T_+} \times \pi_n \mathsf{Ran}(M_*)_{T_+} \to \pi_n \mathsf{Ran}(M_*)_{T_+}$$

Note that taking the union of $W \in \mathsf{Ran}(M_*)_{T_+}$ with itself returns W so that ϕ composed with the diagonal map is th identity. Let $1 \in \pi_n \mathsf{Ran}(M_*)_{T_+}$ be the constant path at T_+. Then 1 is a unit for the ϕ multiplication. Thus

$$\eta = \phi(\eta, \eta) = \eta^2$$

and hence $\eta = 1$. □

Lemma 3.3 (Beilinson-Drinfeld). *The Ran space of a connected space is contractible.*

For a proof see Theorem 5.5.1.6 of [9].

Define a filtration on $\mathsf{Ran}(M_*)$ by cardinality, by taking $\mathsf{Ran}_{\leq k}(M_*)$ to be the subspace of $\mathsf{Ran}(M_*)$ consisting of those based sets $S_+ \subset M_*$ with $|S| \leq k$. For today, think of $\mathsf{Ran}(M_*)$ as a $\mathbb{Z}_{\geq 0}$-stratified space where the stratification keeps track of the size of the finite subset $S \subset M_*$.

For an actual definition of $\mathsf{Ran}(M_*)$ *as a stratified space* see Definition 3.7.1 of [4]. For the moment you should take away the following to aspects of the topology on the Ran space

- Points can collide in a continuous way.
- Points cannot be added or separated in a continuous way.
- We have a basis for the topology on $\mathsf{Ran}(M_*)$ coming from taking $\mathsf{Ran}(-)$ on a basis for the topology on M_*.

Lemma 3.4. *Let M_* be a zero-pointed manifold with M connected. There is a cofiber sequence*

$$\mathsf{Ran}_{\leq k-1}(M_*) \to \mathsf{Ran}_{\leq k}(M_*) \to \mathsf{Conf}_k(M)^+_{\Sigma_k}$$

We need to assume M is connected since $\mathsf{Ran}(M_*)$ only contains finite subsets $S_+ \to M_*$ that are surjective on π_0 where $\mathsf{Conf}_k(M)$ contains arbitrary finite subsets of size k.

Proof. The cofiber of the inclusion

$$\mathsf{Ran}_{\leq k-1}(M_*) \to \mathsf{Ran}_{\leq k-1}(M_*)$$

is the one-point compactification of the complement $\mathsf{Ran}_{\leq k}(M_*) \setminus \mathsf{Ran}_{\leq k-1}(M_*)$. This complement can be identified with the unordered configuration space $\mathsf{Conf}_k(M)_{\Sigma_k}$. □

4. Disk Categories

We want to use the $\mathsf{Ran}_{\leq \bullet}(M_*)$ filtration to define a filtration $\tau^{\leq \bullet} \int_{M_*}$. We do this by mimicking the previous section and define a cardinality filtration on $\mathcal{D}\mathsf{isk}_{n,+/M_*}$.

To start, we need a relationship between $\mathcal{D}\mathsf{isk}_{n,+/M_*}$ and the Ran space $\mathsf{Ran}(M_*)$. Let U, V, V', W be embedded euclidean spaces in M_*. In $\mathcal{D}\mathsf{isk}_{n,+/M_*}$ we make no requirement that $U \to M_*$ be surjective on π_0, like we do for the Ran space. Also note that in $\mathcal{D}\mathsf{isk}_{n,+/M_*}$ we allow both of the following types of morphisms of disks

- combining disks: $V \sqcup V' \hookrightarrow W$
- adding disks: $U \hookrightarrow U \sqcup V$

Compare this to how points interact in the Ran space. The topology on the Ran space allows points to collide in a continuous way. However, in the Ran space there is not a way to continuously *add* points. This (hopefully) motivates restricting our attention to the following category of disks in M_* with only π_0 *surjective* morphisms.

Let Fin_* denote the category of finite based sets I_+ and based maps. Letting $[M_*]$ denote the based set of connected components of M_*, we can take the over category $(\mathsf{Fin}_*)_{[M_*]}$. Let $(\mathsf{Fin}_*^{\mathrm{surj}})_{[M_*]}$

denote the subcategory of $(\mathsf{Fin}_*)_{[M_*]}$ consisting of surjective maps $I_+ \to [M_*]$ and surjective maps between them.

Note that we have a functor $\mathcal{D}\mathsf{isk}_+(M_*) \to \mathsf{Fin}_*$ given by taking connected components.

Definition 4.1. Define $\mathcal{D}\mathsf{isk}_+^{\mathrm{surj}}(M_*)$ to be the pullback category

$$\begin{array}{ccc} \mathcal{D}\mathsf{isk}_+^{\mathrm{surj}}(M_*) & \longrightarrow & \mathcal{D}\mathsf{isk}_+(M_*) \\ \downarrow & & \downarrow \pi_0 \\ (\mathsf{Fin}_*^{\mathrm{surj}})_{[M_*]} & \longrightarrow & (\mathsf{Fin}_*)_{[M_*]} \end{array}$$

Claim. *The category $\mathcal{D}\mathsf{isk}_+^{\mathrm{surj}}(M_*)$ is the exit path category*

$$\mathcal{D}\mathsf{isk}_+^{\mathrm{surj}}(M_*) \simeq \mathsf{Exit}(\mathsf{Ran}(M_*))$$

We define a disk category analogous to $\mathsf{Ran}(M)_{T_+}$. Let $V \to M$ an object of $\mathcal{D}\mathsf{isk}_+(M_*)$. We can consider the under category $(\mathcal{D}\mathsf{isk}_+(M_*))^{V/}$.[3] Define an ∞-subcategory

$$\mathcal{D}\mathsf{isk}_+(M_*)_V \subset (\mathcal{D}\mathsf{isk}_+(M_*))^{V/}$$

consisting of objects

$$\begin{array}{ccc} V & \xrightarrow{f} & V' \\ \downarrow & \swarrow & \\ M_* & & \end{array}$$

with $f_* : \pi_0(V) \to \pi_0(V')$ injective, and those morphisms

$$\begin{array}{ccccc} & \swarrow & V & \searrow & \\ V' & & \xrightarrow{H} & & V'' \\ & \searrow & M_* & \swarrow & \end{array}$$

with $H_* : \pi_0(V') \to \pi_0 * (V'')$ surjective.

4.1. **Cardinality Filtration.** Let $\mathsf{Fin}_*^{\leq k}$ denote the full subcategory of $\mathsf{Fin}_*^{\mathrm{surj}}$ denote the full subcategory consisting of finite based sets I_+ with $|I| \leq k$.

Definition 4.2. Let k be a finite cardinality and M_* a zero-pointed n-manifold. Let $\mathcal{D}\mathsf{isk}_+^{\leq k}(M_*)$ be the pullback category

$$\begin{array}{ccc} \mathcal{D}\mathsf{isk}_+^{\leq k}(M_*) & \longrightarrow & \mathcal{D}\mathsf{isk}_+^{\mathrm{surj}}(M_*) \\ \downarrow & & \downarrow \\ (\mathsf{Fin}_*^{\leq k})_{[M_*]} & \longrightarrow & (\mathsf{Fin}_*^{\mathrm{surj}})_{[M_*]} \end{array}$$

[3] In [2], the definition is given as a subcategory of $\mathcal{D}\mathsf{isk}(\mathcal{B}\mathsf{sc}_{/M_*})$. The only difference between $\mathcal{D}\mathsf{isk}(\mathcal{B}\mathsf{sc}_{/M_*})$ and $\mathcal{D}\mathsf{isk}_+(M_*)$ is whether the basics contain the base point $*$ in their image. Since we are taking things under $V \hookrightarrow M_*$ which contains $*$ in its image, this difference should not matter...I think.

Example. For $k = 1$, the category $\mathcal{D}\mathsf{isk}_{+}^{\leq 1}(X)$ consists of single basics in X that contain a the point. Compare this to $\mathsf{Entr}(X) = \mathcal{B}\mathsf{sc}_{/X}$.

We define the filtration on factorization homology that we sought in (1).

Definition 4.3. For a finite cardinality k, a zero-pointed n-manifold M_* and an augmented n-disk algebra A, define

$$\tau^{\leq k} \int_{M_*} A = \mathsf{colim}\left(\mathcal{D}\mathsf{isk}_{+}^{\leq k}(M_*) \to \mathcal{D}\mathsf{isk}_{+}^{\mathrm{surj}}(M_*) \to \mathcal{D}\mathsf{isk}_{n,+/M_*} \xrightarrow{A} \mathcal{V}\right)$$

We refer to the resulting filtration

$$\tau^{\leq k-1} \int_{M_*} A \to \tau^{\leq k} \int_{M_*} A \to \cdots \to \int_{M_*} A$$

as the *cardinality filtration* on factorization homology.

We relate the cardinality filtration on factorization homology to the cardinality filtration on the Ran space. The following is Corollary 2.3.1 of [2].

Theorem 4.4. *We have an equivalence of categories*

$$\mathcal{D}\mathsf{isk}_{+}^{\leq k}(M_*) \xrightarrow{\sim} \mathcal{D}\mathsf{isk}_{+}^{\leq 1}(\mathsf{Ran}_{\leq k}(M_*))$$

In the limiting case, we obtain an equivalence

$$B\mathcal{D}\mathsf{isk}_{+}(M_*) \xrightarrow{\sim} \mathsf{Ran}(M_*)$$

For $V \to M_*$ in $\mathcal{D}\mathsf{isk}_{+}^{\leq k}(M_*)$, let $[V]$ is a finite based subset of M_* with exactly one point in each connected component of the image of V. We have

$$\mathcal{D}\mathsf{isk}_{+}^{\leq k}(M_*)_V \xrightarrow{\sim} \mathcal{D}\mathsf{isk}_{+}^{\leq 1}(\mathsf{Ran}_{\leq k}(M_*)_{[V]})$$

Idea of Proof. By Proposition 3.7.5 of [4], we have a map

$$\mathsf{Ran}_{\leq k}(-) : \mathcal{D}\mathsf{isk}_{+}^{\leq k}(M_*) \to \mathcal{D}\mathsf{isk}_{+}^{\leq 1}(\mathsf{Ran}_{\leq k}(M_*))$$

Heuristically, this says that $\mathsf{Ran}_{\leq k}$ of a disjoint union of n-disks in M_* looks like a disjoint union of kn-disks in $\mathsf{Ran}_{\leq k}(M_*)$. Saying that $\mathsf{Ran}_{\leq k}$ induces an equivalence of categories is saying that every basic of $\mathsf{Ran}_{\leq k}(M_*)$ comes from a basic in M_*. Note that this is essentially how we defined the topology on $\mathsf{Ran}_{\leq k}(M_*)$. Open sets in $\mathsf{Ran}_{\leq k}(M_*)$ are given by $\mathsf{Ran}_{\leq k}$ of disjoint unions of opens in M_*. □

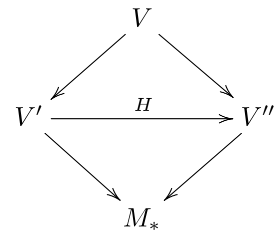
Corollary 4.5. *Let $V \hookrightarrow M_*$ in $\mathcal{D}\mathsf{isk}_{+}^{\mathrm{surj}}(M_*)$. We have an equivalence*

$$B\mathcal{D}\mathsf{isk}_{+}(M_*)_V \simeq \mathsf{Ran}(M_*)_{[V]}$$

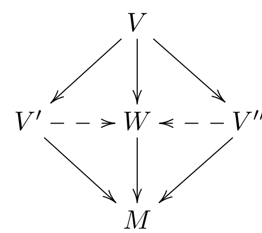
Remark. Corollary 4.5 is not true at the level of ordinary categories. Consider the ordinary category $\mathsf{Disk}_{+}(M_*)_V$ of triangles

$$\begin{array}{ccc} V & \xrightarrow{f} & V' \\ \downarrow & \swarrow & \\ M_* & & \end{array}$$

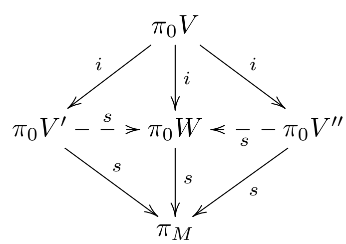
with $f_* : \pi_0(V) \to \pi_0(V')$ injective, and morphisms



with $H_* : \pi_0(V') \to \pi_0 * (V'')$ surjective. By Lemma 3.2, the equivalence would imply $\mathsf{Disk}_+(M_*)_V$ was contractible. Normally, we show disk categories are contractible by showing they are (co)filtered. We show that this category is not filtered. Being filtered would mean that we could always find an object W and dashed maps making the following diagram commute



The corresponding diagram on connected components looks like



For simplicity, take $\pi_0 V$ and $\pi_0 V''$ to have one element and $\pi_0 V'$ to have two elements. Since 1 surjects onto $\pi_0 W$, we have to have $\pi_0 W = 1$. But now $1 \to 2 \to 1$ is not injective so the upper left triangle cannot commute. A similar argument shows that $\mathsf{Disk}_+(M_*)_V$ is not cofiltered.

5. Layer Computations

The point of this section is to prove the following result

Theorem 5.1. *Let M_* be a zero-pointed n-manifold with M connected. There is a cofiber sequence*

$$\tau^{\leq k-1} \int_{M_*} A \to \tau^{\leq k} \int_{M_*} A \to \mathsf{Conf}_k^{\mathrm{fr},\neg}(M_*) \bigotimes_{\Sigma_k \wr O(n)} \mathsf{cKer}^{\mathrm{aug}} A(\mathbb{R}^n_+)^{\otimes k}$$

Before we prove this, we need to address the problem raised in Andy's talk that $\mathsf{Conf}_k(M_*)$ may not be a zero-pointed manifold if $*$ is not isolated. To fix this, we introduce a new space, homotopy equivalent to $\mathsf{Conf}_k(M_*)$ which is always is a zero-pointed manifold. The following is Lemma 1.2.3 of [2].

Lemma 5.2. *Let i be a finite cardinality. For each zero-pointed manifold M_* there are Σ_i-zero-pointed manifolds with corners $C_i(M_*)$ and $C_i^{\neg}(M_*)$ such that*

1) There are Σ_i-equivariant based homotopy equivalences

$$\mathsf{Conf}_i(M_*) \simeq C_i(M_*)$$

and

$$\mathsf{Conf}_i^{\neg}(M_*) \simeq C_i^{\neg}(M_*)$$

2) There are Σ_i-equivariant isomorphisms between zero-pointed manifolds with corners:

$$C_i(M_*)^{\neg} \simeq C_i^{\neg}(M_*^{\neg})$$

We prove the analogue of Lemma 3.4 for the corresponding disk categories. This is Lemma 2.3.3 of [2].

Lemma 5.3. *Let i be a finite cardinaility, and let M_* be a zero-pointed manifold with M connected. There is a canonical cofiber sequence among ∞-categories*

$$\mathcal{D}\mathsf{isk}_+^{<i}(M_*) \to \mathcal{D}\mathsf{isk}_+^{\leq i}(M_*) \to \mathcal{D}\mathsf{isk}_+^{\leq 1}(C_i^{\neg}(M_*)_{\Sigma_i})$$

Remark. Note that by Lemma 4.4, we can rewrite this cofiber sequence as

$$\mathcal{D}\mathsf{isk}_+^{\leq 1}(\mathsf{Ran}_{\leq i-1}(M_*)) \to \mathcal{D}\mathsf{isk}_+^{\leq 1}(\mathsf{Ran}_{\leq i}(M_*)) \to \mathcal{D}\mathsf{isk}_+^{\leq 1}(C_i^{\neg}(M_*)_{\Sigma_i})$$

This is the form of the cofiber sequence we will prove exists. Taking $M_* = M^+$ we can identify $C_i^{\neg}(M_*)_{\Sigma_i}$ with $\mathsf{Conf}_i(M)^+_{\Sigma_i}$. Ignoring the $\mathcal{D}\mathsf{isk}_+^{\leq 1}$ for a moment, we arrive at the cofiber sequence of Lemma 3.4,

$$\mathsf{Ran}_{\leq i-1}(M_*) \to \mathsf{Ran}_{\leq i}(M_*) \to \mathsf{Conf}_k(M)^+_{\Sigma_i}$$

The proof of the cofiber sequence for disk categories will work a lot like the proof of Lemma 3.4. We know the complement $\mathsf{Ran}_{\leq i}(M_*) \setminus \mathsf{Ran}_{\leq i-1}(M_*)$ is $\mathsf{Conf}_i(M)_{\Sigma_i}$ and that the cofiber should be something like the one-point compactification of this complement. Since we want a cofiber sequence on the level $\mathcal{D}\mathsf{isk}_+^{\leq 1}$ categories, we have to do a lot more technical work with stratified spaces and enter/exit path categories.

We will need the following construction which can be found in Definition 7.3.3 or 7.3.11 of [4]. *"Construction"*. Let $Y \to X$ be a conically smooth embedding which is proper and constructible. The *link* of Y in X is a stratified space $\mathsf{Link}_Y X$ equipped with a proper, constructible bundle

$$\pi_Y \colon \mathsf{Link}_Y X \to Y$$

In the case that X is a smooth manifold and Y is a properly embedded smooth submanifold of X, the link of Y in X is the sphere bundle of a normal bundle of Y in X.

Because I don not have time to actually define the link, we are going to pretend the link is just a sphere bundle of a normal bundle when it comes up.

We also need the notion of "finitary stratified spaces" which is covered in §8.3 of [4]. In differential topology, it is useful to work with manifolds that have handlebody decompositions. A handlebody decomposition is a way of building a manifold out of disks glued together along their boundaries in a nice way. In the stratified world, disks are replaced by basics and these gluings are replaced by collar-gluings.

Definition 5.4 (Pretend Definition)**.** The category of *finitary stratified spaces* is the smallest ∞-subcategory of stratified spaces that contains Bsc and is closed under the formation of collar-gluings.

Warning. I don't fully understand the proof of Lemma 2.3.3 of [2] and therefore almost assuredly made mistakes in trying to explain it below.

Proof of Lemma 5.3. By Corollary 4.4, it suffices to identify the cofiber of

$$\mathcal{D}\mathsf{isk}_{+}^{\leq 1}(\mathsf{Ran}_{<i}(M_*)) \to \mathcal{D}\mathsf{isk}_{+}^{\leq 1}(\mathsf{Ran}_{\leq i}(M_*))$$

Let L be the link $L := \mathsf{Link}_{\mathsf{Ran}_{<i}(M_*)}\mathsf{Ran}_{\leq i}(M_*)$. Consider the blow-up

$$\mathsf{Bl}_{\mathsf{Ran}_{<i}(M_*)}(\mathsf{Ran}_{\leq i}(M_*)) \to \mathsf{Ran}_{\leq i}(M_*)$$

Note that this blow-up is a smooth manifold with corners whose boundary is the sphere bundle of the normal bundle of $\mathsf{Ran}_{<i}(M_*) \subset \mathsf{Ran}_{\leq i}(M_*)$; i.e., the link L. By Lemma 3.7.6 of [4], the resulting diagram

$$\begin{array}{ccc} L & \longrightarrow & \mathsf{Bl}_{\mathsf{Ran}_{<i}(M_*)}(\mathsf{Ran}_{\leq i}(M_*)) \\ \downarrow & & \downarrow \\ \mathsf{Ran}_{<i}(M_*) & \longrightarrow & \mathsf{Ran}_{\leq i}(M_*) \end{array}$$

is a pushout diagram of stratified spaces. We would like to replace the upper-right corner with the $\mathsf{Conf}_i(M)_{\Sigma_i}$. Note that on the level of stratified spaces, we can factor the top horizontal map by choosing a collar-neighborhoood

$$\begin{array}{ccc} & L & \\ {}^{0}\swarrow & & \searrow \\ L \times [0,1) & \dashrightarrow & \mathsf{Bl}_{\mathsf{Ran}_{<i}(M_*)}(\mathsf{Ran}_{\leq i}(M_*)) \end{array}$$

where $[0,1)$ is stratified over $\{0 < 1\}$ with 0-stratum $\{0\}$ and 1-stratum $(0,1)$.

Consider the pullback diagram

$$\begin{array}{ccc} \mathsf{Bl}_{\mathsf{Ran}_{<i}(M_*)}(\mathsf{Ran}_{\leq i}(M_*))|_{\mathsf{Conf}_i(M)_{\Sigma_i}} & \longrightarrow & \mathsf{Bl}_{\mathsf{Ran}_{<i}(M_*)}(\mathsf{Ran}_{\leq i}(M_*)) \\ \downarrow & & \downarrow \\ \mathsf{Conf}_i(M)_{\Sigma_i} & \longrightarrow & \mathsf{Ran}_{\leq i}(M_*) \end{array}$$

Since blow-ups are isomorphisms away from the explosion, the left vertical arrow is an isomorphism. The top horizontal map is the inclusion of the interior of a smooth manifold with corners. By Theorem 8.3.10(1) of [4], the interior of a compact smooth manifold with corners is finitary. Thus $\mathsf{Conf}_i(M)_{\Sigma_i}$ can be viewed as a finitary stratified space.

By Example 8.3.4 of [4], we have a collar-gluing

$$\begin{array}{ccc} L \times (0,1) & \longrightarrow & \mathsf{Bl}_{\mathsf{Ran}_{<i}(M_*)}(\mathsf{Ran}_{\leq i}(M_*))|_{\mathsf{Conf}_i(M)_{\Sigma_i}} \\ \downarrow & & \downarrow \\ L \times [0,1) & \longrightarrow & \mathsf{Bl}_{\mathsf{Ran}_{<i}(M_*)}(\mathsf{Ran}_{\leq i}(M_*)) \end{array}$$

Using the above, we can replace the space in the upper right-hand corner with $\mathsf{Conf}_i(M)_{\Sigma_i}$. By Lemma 6.3.1 of [4], collar-gluings induce pushout diagrams upon taking enter path categories. Thus we have a colimit diagram

$$\begin{array}{ccc} \mathsf{Entr}(L \times (0,1)) & \longrightarrow & \mathsf{Entr}(\mathsf{Conf}_i(M_*)) \\ \downarrow & & \downarrow \\ \mathsf{Entr}(L \times [0,1)) & \longrightarrow & \mathsf{Entr}(\mathsf{Bl}_{\mathsf{Ran}_{<i}(M_*)}(\mathsf{Ran}_{\leq i}(M_*))) \end{array}$$

We have a colimit diagram in stratified spaces

$$\begin{array}{ccccc} & & L \times (0,1) & \longrightarrow & \mathsf{Conf}_i(M)_{\Sigma_i} \\ & & \downarrow & & \downarrow \\ L \times \{0\} & \longrightarrow & L \times [0,1] & & \\ \downarrow & & & \searrow & \downarrow \\ \mathsf{Ran}_{<i}(M_*) & & \longrightarrow & & \mathsf{Ran}_{\leq i}(M_*) \end{array}$$

which witnesses $\mathsf{Ran}_{\leq i}(M_*)$ as a collar-gluing of $\mathsf{Ran}_{<i}(M_*)$ and the finitary stratified space $\mathsf{Conf}_i(M)_{\Sigma_i}$. Since finitary stratified spaces are closed under collar-gluings (by definition), we inductively get that $\mathsf{Ran}_{\leq i}(M_*)$ is a finitary stratified space.

The finitary condition allows us to apply Lemma 3.4 of [5] (which makes use of Lemma 6.3.1 of [4]) to say we get a colimit diagram on the level of enter-path categories. Putting together these two colimit diagrams of enter-path categories, we have a colimit diagram

$$\begin{array}{ccccc} & & \mathsf{Entr}(L) & \longrightarrow & \mathsf{Entr}(\mathsf{Conf}_i(M)_{\Sigma_i}) \\ & & \downarrow & & \downarrow \\ \mathsf{Entr}(L) & \longrightarrow & \mathsf{Entr}(L) \times [1] & & \\ \downarrow & & & \searrow & \downarrow \\ \mathsf{Entr}(\mathsf{Ran}_{<i}(M_*)) & & \longrightarrow & & \mathsf{Entr}(\mathsf{Ran}_{\leq i}(M_*)) \end{array}$$

and that we can identify the cofiber of

$$\mathsf{Entr}(\mathsf{Ran}_{<i}(M_*)) \to \mathsf{Entr}(\mathsf{Ran}_{\leq i}(M_*))$$

with the cofiber

$$\mathsf{Entr}(L) \to \mathsf{Entr}(\mathsf{Conf}_i(M)_{\Sigma_i}) \to \mathsf{Entr}(L)^{\triangleleft} \coprod_{\mathsf{Entr}(L)} \mathsf{Entr}(\mathsf{Conf}_i M_{\Sigma_i})$$

Here $\mathsf{Entr}(X)^{\triangleleft}$ denotes the ∞-category obtained by adjoining an initial vertex to $\mathsf{Entr}(X)$. By Lemma 6.1.4 of [4], we have

$$\mathsf{Entr}(L)^{\triangleright} \simeq \mathsf{Entr}(C(L))$$

where $C(L)$ is the open cone on L. In $C(L)$, the cone point is a minimal element. Let $L^{\triangleleft}$ be $C(L)$ but stratified so that the cone point is a maximal element. Then

$$\mathsf{Entr}(L)^{\triangleleft} \simeq \mathsf{Entr}(L^{\triangleleft})$$

Now by Proposition 1.2.13 of [4], we get

$$\mathsf{Entr}(L^{\triangleleft}) \coprod_{\mathsf{Entr}(L)} \mathsf{Entr}(\mathsf{Conf}_i M_{\Sigma_i}) \simeq L^{\triangleleft} \coprod_{L} \mathsf{Conf}_i(M)_{\Sigma_i}$$

The right hand side looks like the one-point compactification of the unordered configuration space.

Finally, by "direct inspection,"[4] we have

$$L^{\triangleleft} \coprod_{L} \mathsf{Conf}_i(M)_{\Sigma_i} \xrightarrow{\sim} \mathcal{D}\mathsf{isk}_{+}^{\leq 1}(C_i^{\neg}(M_*))$$

□

[4]This is in quotes since we did not define $C_i^{\neg}(M_*)$ during this talk.

5.1. **Reduced Extensions.** We have identified the quotient category

$$\mathcal{D}\mathsf{isk}_{+}^{\leq i}(M_*)/\mathcal{D}\mathsf{isk}_{+}^{<i}(M_*)$$

To prove Theorem 5.1, we need to understand what happens to an n-disk algebra A when we pass to this quotient category.

Let $f : \mathcal{K} \to \mathcal{K}'$ be a functor of small ∞-categories. Let $\mathcal{V}$ be a presentable ∞-category. There is an adjunction among functor ∞-categories

$$f_! : \mathsf{Fun}(\mathcal{K}, \mathcal{V}) \rightleftarrows \mathsf{Fun}(\mathcal{K}', \mathcal{V}) : f^*$$

where f^* is precomposition and $f_!$ is left Kan extension. Assume additionally that $\mathcal{V}$ has a zero object and that $\mathcal{K}$ is a pointed ∞-category $x\colon * \to \mathcal{K}$. Here a pointed category is a category with a distinguished object that may not be a zero object. Define a full ∞-subcategory $\mathsf{Fun}_0(\mathcal{K}, \mathcal{V})$ as the fiber over the zero object

$$\mathsf{Fun}_0(\mathcal{K}, \mathcal{V}) \to \mathsf{Fun}(\mathcal{K}, \mathcal{V}) \xrightarrow{x^*} \mathcal{V}$$

Refer to functors in $\mathsf{Fun}_0(\mathcal{K}, \mathcal{V})$ as *reduced*. The following is Lemma 2.3.4 of [2].

Lemma 5.5. *The inclusion* $\mathsf{Fun}_0(\mathcal{K}, \mathcal{V}) \to \mathsf{Fun}(\mathcal{K}, \mathcal{V})$ *admits a left adjoint*

$$(-)^{\mathrm{red}} : \mathsf{Fun}(\mathcal{K}, \mathcal{V}) \to \mathsf{Fun}_0(\mathcal{K}, \mathcal{V})$$

This left adjoint fits into a cofiber sequence in $\mathsf{Fun}(\mathcal{K}, \mathcal{V})$*,*

$$x_! x^* \to \mathsf{id} \to (-)^{\mathrm{red}}$$

Add the assumption that $\mathcal{V}$ is stable. The following is Lemma 2.3.5 of [2].

Lemma 5.6. *Let* $i : \mathcal{K}_0 \to \mathcal{K}$ *be a fully faithful functor among ∞-categories. Consider the functor* $j : \mathcal{K} \to \mathcal{K}/\mathcal{K}_0 := * \sqcup_{\mathcal{K}_0} \mathcal{K}$ *to the cone, regarded as a pointed ∞-category. There is a cofiber sequence in* $\mathsf{Fun}(\mathcal{K}, \mathcal{V})$*,*

$$i_! i^* \to \mathsf{id} \to j^* j_!^{\mathrm{red}}$$

Assume that the point $x : * \to \mathcal{K}$ factors through the fully faithful functor $i : \mathcal{K} \to \mathcal{K}_0$. In this case, we have a commutative diagram

$$\mathcal{K} \xrightarrow{j} \mathcal{K}/\mathcal{K}_0$$

where $y(*) = *$ is the point in $\mathcal{K}/\mathcal{K}_0$.

Lemma 5.7. *For any* $F \in \mathsf{Fun}_0(\mathcal{K}/\mathcal{K}_0, \mathcal{V})$*, there is an equivlance*

$$\mathsf{colim}(\mathcal{K}/\mathcal{K}_0 \xrightarrow{F} \mathcal{V}) \simeq \mathsf{colim}(\mathcal{K} \xrightarrow{j^* F} \mathcal{V})$$

in $\mathcal{V}$*.*

Proof. The colimit

$$\mathsf{colim}(\mathcal{K}/\mathcal{K}_0 \xrightarrow{j_!^{\mathrm{red}} F} \mathcal{V})$$

is taken over a pushout category,

$$\begin{array}{ccc} \mathcal{K}_0 & \longrightarrow & \mathcal{K} \\ \downarrow & & \downarrow \\ * & \xrightarrow[y]{} & \mathcal{K}/\mathcal{K}_0 \end{array}$$

By Proposition 4.4.2.2 of [10], the colimit of $j_!^{\mathrm{red}}F$ over $\mathcal{K}/\mathcal{K}_0$ may be identified with the pushout of the colimits,

$$\begin{array}{ccc} \operatorname{colim}_{\mathcal{K}_0}\left(i^*j^*j_!^{\mathrm{red}}F\right) & \longrightarrow & \operatorname{colim}_{\mathcal{K}}\left(j^*j_!^{\mathrm{red}}F\right) \\ \downarrow & & \downarrow \\ \operatorname{colim}_{*}\left(y^*j_!^{\mathrm{red}}F\right) & \longrightarrow & \operatorname{colim}_{\mathcal{K}/\mathcal{K}_0}\left(j_!^{\mathrm{red}}F\right) \end{array}$$

Since $j_!^{\mathrm{red}}F$ is reduced, $y^*j_!^{\mathrm{red}}F$ is the zero functor. Thus the bottom left colimit is $0 \in \mathcal{V}$. Since $j \circ i$ factors through the base point $y : * \to \mathcal{K}/\mathcal{K}_0$, we have $i^*j^*i_!^{\mathrm{red}}F$ equivalent to the zero functor. Thus the upper left colimit is $0 \in \mathcal{V}$. Thus the right vertical arrow is an equivalence. □

Proof of Theorem 5.1. Let A be an augmented n-disk algebra. Let

$$i_k^*A\colon \mathcal{D}\mathsf{isk}_+^{\leq k}(M_*) \to \mathcal{V}$$

be the restriction of A to $\mathcal{D}\mathsf{isk}_+^{\leq k}(M_*)$. Taking i to be the inclusion $i_{k-1}\colon \mathcal{D}\mathsf{isk}_+^{\leq k-1}(M_*) \to \mathcal{D}\mathsf{isk}_+^{\leq k}(M_*)$ in Lemma 5.6, we get a cofiber sequence

$$(i_{k-1})_!i_{k-1}^*i_k^*A \to i_k^*A \to j^*j_!^{\mathrm{red}}i_k^*A$$

By definition,

$$\operatorname{colim}\left(\mathcal{D}\mathsf{isk}_+^{\leq k}(M_*) \xrightarrow{i_k^*A} \mathcal{V}\right) = \tau^{\leq k}\int_{M_*} A$$

We have that $i_{k-1}^*i_k^*A$ is the restriction of A to $\mathcal{D}\mathsf{isk}_+^{\leq k-1}(M_*)$. Thus

$$\operatorname{colim}\left(\mathcal{D}\mathsf{isk}_+^{\leq k-1}(M_*) \xrightarrow{i_{k-1}^*i_k^*A} \mathcal{V}\right) = \tau^{\leq k-1}\int_{M_*} A$$

Since $(i_{k-1})_!$ is a left Kan extension, we can compute the colimit of $(i_{k-1})_!i_{k-1}^*i_k^*A$ over the category $\mathcal{D}\mathsf{isk}_+^{\leq k-1}(M_*)$,

$$\operatorname{colim}\left(\mathcal{D}\mathsf{isk}_+^{\leq k}(M_*) \xrightarrow{(i_{k-1})_!i_{k-1}^*i_k^*A} \mathcal{V}\right) = \operatorname{colim}\left(\mathcal{D}\mathsf{isk}_+^{\leq k-1}(M_*) \xrightarrow{i_{k-1}^*i_k^*A} \mathcal{V}\right)$$

Finally, by Lemma 5.7, we have an equivalence

$$\operatorname{colim}\left(\mathcal{D}\mathsf{isk}_+^{\leq k}(M_*) \xrightarrow{j^*j_!^{\mathrm{red}}i_k^*A} \mathcal{V}\right) \simeq \operatorname{colim}\left(\mathcal{D}\mathsf{isk}_+^{\leq k}(M_*)/\mathcal{D}\mathsf{isk}_+^{\leq k-1}(M_*) \xrightarrow{j_!^{\mathrm{red}}i_k^*A} \mathcal{V}\right)$$

Thus we get a cofiber sequence in $\mathcal{V}$,

$$\tau^{\leq k-1}\int_{M_*} A \to \tau^{\leq k}\int_{M_*} A \to \operatorname{colim}\left(\mathcal{D}\mathsf{isk}_+^{\leq k}(M_*)/\mathcal{D}\mathsf{isk}_+^{\leq k-1}(M_*) \xrightarrow{j_!^{\mathrm{red}}i_k^*A} \mathcal{V}\right)$$

By Lemma 5.3, the colimit on the right hand side can be identified with

$$\operatorname{colim}\left(\mathcal{D}\mathsf{isk}_+^{\leq 1}(C_i^{\neg}(M_*)) \xrightarrow{A(\mathbb{R}^n)^{\otimes k}} \mathcal{V}\right)$$

By Lemma 5.2 we can identify this colimit as

$$\mathsf{Conf}_k^{\mathrm{fr},\neg}(M_*) \bigotimes_{\Sigma_k \wr O(n)} \mathsf{cKer}^{\mathrm{aug}} A(\mathbb{R}^n)^{\otimes k}$$

The theorem follows. □

6. Convergence

We study the question of when the cardinality filtration converges.

The following is Lemma 2.3.2 in [2].

Theorem 6.1. *The functor*

$$\mathcal{D}\mathsf{isk}_{+}^{\mathrm{surj}}(M_*) \to \mathcal{D}\mathsf{isk}_{+}(M_*)$$

is final.

Remark. If we try to work in the non-infinity world, the analogous result is not true for the corresponding posets. The result fails since we will use Corollary 4.5, which is false on the level of posets, as Remarked after Corollary 4.5.

The outline of the proof is as follows:

Step 1. Reduce to M connected as follows. Writing M_* as a wedge, we can view the functor as a product of functors. Since the product of final functors is final, it suffices to assume M is connected.

Step 2. Use Quillen's theorem A to restate finality in terms of contractibility of

$$\mathcal{D}\mathsf{isk}_{+}^{\mathrm{surj}}(M_*) \times_{\mathcal{D}\mathsf{isk}_{+}(M_*)} \left(\mathcal{D}\mathsf{isk}_{+}(M_*)\right)^{V/}$$

for every V in $\mathcal{D}\mathsf{isk}_{+}(M_*)$.

Step 3. Construct an adjunction

$$\mathcal{D}\mathsf{isk}_{+}(M_*)_V \leftrightarrows \mathcal{D}\mathsf{isk}_{+}^{\mathrm{surj}}(M_*) \times_{\mathcal{D}\mathsf{isk}_{+}(M_*)} \left(\mathcal{D}\mathsf{isk}_{+}(M_*)\right)^{V/}$$

Step 4. Classifying spaces of a pair of categories in an adjunction are the same,

$$B\mathcal{D}\mathsf{isk}_{+}(M_*)_V \simeq B\left(\mathcal{D}\mathsf{isk}_{+}^{\mathrm{surj}}(M_*) \times_{\mathcal{D}\mathsf{isk}_{+}(M_*)} \left(\mathcal{D}\mathsf{isk}_{+}(M_*)\right)^{V/}\right)$$

Step 5. To show that $\mathcal{D}\mathsf{isk}_{+}(M_*)_V$ is contractible, use Lemma 4.5 and Lemma 3.2,

$$B\mathcal{D}\mathsf{isk}_{+}(M_*)_V \simeq \mathsf{Ran}(M_*)_{[V]} \simeq \mathrm{pt}$$

We construct an adjunction

$$f : \mathcal{D}\mathsf{isk}_{+}(M_*)_V \rightleftarrows \mathcal{D}\mathsf{isk}_{+}^{\mathrm{surj}}(M_*) \times_{\mathcal{D}\mathsf{isk}_{+}(M_*)} \left(\mathcal{D}\mathsf{isk}_{+}(M_*)\right)^{V/} : g$$

Define f by using the inclusion morphisms

$$\mathcal{D}\mathsf{isk}_{+}(M_*)_V \subset \left(\mathcal{D}\mathsf{isk}_{+}(M_*)\right)^{V/}$$

and

$$\mathcal{D}\mathsf{isk}_{+}(M_*)_V \to \mathcal{D}\mathsf{isk}_{+}^{\mathrm{surj}}(M_*)$$

Define g on an object $V \to V' \to M_*$ by forgetting the components of V' for which $\pi_0 V \to \pi_0 V'$ is not injective.

Corollary 6.2. *The cardinality filtration on factorization homology always converges.*

Proof. Since $\mathcal{D}\mathsf{isk}_{+}^{\mathrm{surj}}(M_*) \to \mathcal{D}\mathsf{isk}_{+}(M_*)$ and $\mathcal{D}\mathsf{isk}_{+}(M_*) \to \mathcal{D}\mathsf{isk}_{n,+/M_*}$ are both final functors, we have

$$\int_{M_*} A \simeq \mathsf{colim}\left(\mathcal{D}\mathsf{isk}_{+}^{\mathrm{surj}}(M_*) \xrightarrow{A} \mathcal{V}\right)$$

The ∞-category $\mathcal{D}\mathsf{isk}_{+}^{\mathrm{surj}}(M_*)$ is a sequential colimit of the ∞-subcategories $\mathcal{D}\mathsf{isk}_{+}^{\leq k}(M_*)$. Thus

$$\begin{aligned}\int_{M_*} A &\simeq \mathsf{colim}\left(\left(\mathsf{colim}_{k}\mathcal{D}\mathsf{isk}_{+}^{\leq k}(M_*)\right) \xrightarrow{A} \mathcal{V}\right)\\ &\simeq \mathsf{colim}_{k}\left(\mathsf{colim}\left(\mathcal{D}\mathsf{isk}_{+}^{\leq k}(M_*) \xrightarrow{A} \mathcal{V}\right)\right)\\ &= \mathsf{colim}_{k}\left(\tau^{\leq k}\int_{M_*} A\right)\end{aligned}$$

□

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