# Nonabelian Poincaré Duality

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**Abstract**

We sketch Ayala and Francis' proof of Nonabelian Poincaré Duality from [1]. Nonabelian Poincaré Duality follows easily from the equivalence between n-disk algebras and homology theories for manifolds stated last time, so we'll really sketch a proof of this equivalence. The main ingredient that we'll assume is that factorization homology satisfies ⊗-excision (hence defines a homology theory for manifolds). If time permits, we'll also discuss why factorization homology satisfies ⊗-excision.

## Contents



# Part I
# Nonabelian Poincaré Duality

## 1 Nonabelian Poincaré Duality from the classification theorem

In this section we show how Nonabelian Poincaré Duality follows from the classification of homology theories for manifolds in terms of factorization homology stated last time

(Theorem 1.4). First we recall some notation.

**1.1 Notation.** For a nonnegative integer $n$, write $\mathrm{Mfld}_n$ for the 1-category of $n$-manifolds and embeddings. Write $\mathrm{Disk}_n \subset \mathrm{Mfld}_n$ for the full subcategory spanned by those $n$-manifolds isomorphic to a finite disjoint union of Euclidean spaces. Both $\mathrm{Disk}_n$ and $\mathrm{Mfld}_n$ have symmetric monoidal structures given by disjoint union.

We write $\mathbf{Mfld}_n$ for the $\infty$-category associated to the topological category[1] with objects $n$-manifolds and morphism spaces the spaces $\mathrm{Emb}(M, N)$ of embeddings $M \hookrightarrow N$ with the compact-open topology. We write $\mathbf{Disk}_n \subset \mathbf{Mfld}_n$ for the full subcategory spanned by those $n$-manfiolds isomorphic toa finite disjoint union of Euclidean spaces.

**1.2 Definition.** A symmetric monoidal $\infty$-category $(V, \otimes)$ is ***$\otimes$-presentable*** if the following conditions are satisfied:

(1.2.1) The underlying $\infty$-category $V$ is presentable.

(1.2.2) The tensor product functor $\otimes \colon V \times V \to V$ preserves colimits separately in each variable.

**1.3 Remark.** Many use the term *presentably symmetric monoidal* in lieu of *$\otimes$-presentable* [2]. We'll use the terminology of Ayala–Francis in this talk.

**1.4 Theorem** ([1, Theorem 3.24])**.** *Let $V$ be $\otimes$-presentable symmetric monoidal $\infty$-category. Then there is an equivalence of $\infty$-categories*

$$\int \colon \mathbf{Alg}_n(V) \rightleftarrows \mathbf{H}(\mathbf{Mfld}_n, V) : \mathrm{ev}_{\mathbf{R}^n} .$$

For Nonabelian Poincaré Duality, we'll mostly be interested in the case that $V$ is the $\infty$-category of spaces with the cartesian monoidal structure.

**1.5 Definition.** Let $X$ be a topolocial space and $(Z, z_0)$ a pointed topological space. The ***support*** of a map $f \colon X \to Z$ is the subspace

$$\mathrm{supp}(f) \coloneqq \{\, x \in X \mid f(x) \neq z_0 \,\} .$$

We say that $f$ is ***compactly supported*** if the closure of $\mathrm{supp}(f)$ is compact, and write

$$\mathrm{Map}_c(X, Z) \subset \mathrm{Map}(X, Z)$$

for the subspace of compactly supported maps.

**1.6 Theorem** (Nonabelian Poincaré Duality)**.** *Let $M$ be an $n$-manifold and $Z$ an $n$-connective pointed topological space. Then the natural map*

$$\int_M \mathrm{Map}_c(-, Z) \to \mathrm{Map}_c(M, Z)$$

*is an equivalence in* $\mathbf{Spc}$.

[1] As a quasicategory, is the simplicial nerve of the fibrant simplicial category obtained from the topological category by applying the singular simplicial set functor $\mathrm{Sing} \colon \mathbf{Top} \to s\mathbf{Set}$.

**1.7 Nonexample.** If $Z = S^0$, then for each $U \in \mathbf{Disk}_{n,/M}$ we have

$$\mathrm{Map}_c(U, S^0) = * \,,$$

so

$$\int_M \mathrm{Map}_c(-, S^0) = \operatorname*{colim}_{\mathbf{Disk}_{n,/M}} * \simeq B(\mathbf{Disk}_{n,/M}) \,,$$

where $B(\mathbf{Disk}_{n,/M})$ denotes the classifying space of the $\infty$-category $\mathbf{Disk}_{n,/M}$. In particular, $B(\mathbf{Disk}_{n,/M})$ is connected. On the other hand, if $M$ is compact, then

$$\mathrm{Map}_c(M, S^0) = \mathrm{Map}(M, S^0) \,,$$

and $\mathrm{Map}(M, S^0)$ has at least two connected components. So the condition that $Z$ be $n$-connective cannot be droped.

**1.8 Remark** (on $n$-connectivity)**.** For any pointed space $Z$, we have an equivalence

$$\mathrm{Map}_c(\mathbf{R}^n, Z) \simeq \Omega^n Z \,.$$

We conclude that $\int_M \mathrm{Map}_c(-, Z)$ only depends on the $n$-connective cover $\tau_{\geq n} Z$ of $Z$. However, by choosing $M$ appropriately, one can show that

$$\mathrm{Map}_c(-, Z) \colon \mathbf{Mfld}_n \to \mathbf{Spc}$$

*does* depend on $\tau_{<n} Z$. So the statement of Theorem 1.6 cannot be true without the $n$-connectivity assumption.

Assuming Theorem 1.4, since factorization homology defines a homology theory for $n$-manifolds, the proof of Theorem 1.6 follows once we knowfrom the following two facts:

(1) For any pointed $n$-connective topological space $Z$,

$$\mathrm{Map}_c(-, Z) \colon \mathbf{Mfld}_n \to \mathbf{Spc}$$

defines a homology theory for $n$-manifolds.

(2) Both $\int_{(-)} \mathrm{Map}_c(-, Z)$ and $\mathrm{Map}_c(-, Z)$ take the same value on $\mathbf{R}^n$.

The second point is easy to see:

**1.9 Observation.** Since $\mathbf{R}^n$ is the final object of $\mathbf{Disk}_{n,/\mathbf{R}^n}$, the factorization homology of $A \in \mathbf{Alg}_n(V)$ evaluated on $\mathbf{R}^n$ is given by

$$\int_{\mathbf{R}^n} A \simeq A(\mathbf{R}^n) \,.$$

## 2 Compactly supported maps define a homology theory for manifolds

In this section we'll sketch a proof that for any $n$-connective pointed space $Z$, compactly supported maps

$$\mathrm{Map}_c(-, Z)\colon \mathbf{Mfld}_n \to \mathbf{Spc}$$

defines a homology theory for $n$-manifolds valued in the $\infty$-category of spaces with the cartesian symmetric monoidal structure. So that the statement of Nonabelian Poincaré Duality makes sense, we record the following obvious fact:

**2.1 Lemma.** *Let $Z$ be a pointed space. Then the functor*

$$\mathrm{Map}_c(-, Z)\colon \mathbf{Mfld}_n \to \mathbf{Spc}$$

*is naturally symmetric monoidal, where $\mathbf{Mfld}_n$ is given the disjoint union symmetric monoidal structure and $\mathbf{Spc}$ is given the cartesian symmetric monoidal structure. In particular, the restriction of $\mathrm{Map}_c(-, Z)$ to $\mathbf{Disk}_n$ defines an $n$-disk algebra*

$$\mathrm{Map}_c(-, Z)\colon \mathbf{Disk}_n \to \mathbf{Spc}\,.$$

**2.2 Proposition.** *Let $Z$ be an $n$-connective pointed space. Then for any collar gluing $M \cong M' \cup^{M_0 \times \mathbf{R}} M''$ of $n$-manifolds, the natural map*

$$\mathrm{Map}_c(M', Z) \times_{\mathrm{Map}_c(M_0 \times \mathbf{R}, Z)} \mathrm{Map}_c(M'', Z) \to \mathrm{Map}_c(M, Z)$$

*is an equivalence in $\mathbf{Spc}$.*

*Proof sketch.* Since $M_0 \hookrightarrow M$ is proper, a compactly supported map $M \to Z$ can be restricted to a compactly supported map from $M_0$ (or $M \smallsetminus M'$ or $M \smallsetminus M''$). Thus we have a commuative diagram

$$\begin{tikzcd}
\mathrm{Map}_c(M', Z) \times \mathrm{Map}_c(M'', Z) \arrow[rr] \arrow[dr, dotted] \arrow[dd] & & \mathrm{Map}_c(M'', Z) \arrow[d] \\
& \mathrm{Map}_c(M, Z) \arrow[r] \arrow[d] & \mathrm{Map}_c(M \smallsetminus M'', Z) \arrow[d] \\
\mathrm{Map}_c(M', Z) \arrow[r] & \mathrm{Map}_c(M \smallsetminus M', Z) \arrow[r] & \mathrm{Map}_c(M_0, Z)\,.
\end{tikzcd}$$

(1) Since $M_0 \subset M$ has a regular neighborhood,

$$\mathrm{Map}_c(M \smallsetminus M', Z) \to \mathrm{Map}_c(M_0, Z) \qquad \text{and} \qquad \mathrm{Map}_c(M \smallsetminus M'', Z) \to \mathrm{Map}_c(M_0, Z)$$

are Serre fibrations.

(2) The inner square is a (set-theoretic) pullback because $M \cong M' \cup^{M_0 \times \mathbf{R}} M''$, hence is a homotopy pullback.

(3) By point-set topology, the lower and right sequences are fiber sequences.

(4) There is a right homotopy coherent action of $\Omega\,\mathrm{Map}_c(M_0, Z)$ on $\mathrm{Map}_c(M', Z)$ and a left homotopy coherent action of $\Omega\,\mathrm{Map}_c(M_0, Z)$ on $\mathrm{Map}_c(M'', Z)$ and a map

$$\mathrm{Map}_c(M', Z) \times_{\Omega\,\mathrm{Map}_c(M_0,Z)} \mathrm{Map}_c(M'', Z)\ .$$

Since $Z$ is $n$-connective and $\dim(M_0) = n-1$, the space $\mathrm{Map}_c(M_0, Z)$ is connected. One sees that the above map is an equivalence. The result follows from the identification

$$\mathrm{Map}_c(M_0 \times \mathbf{R}, Z) \simeq \mathrm{Map}_c(\mathbf{R}, \mathrm{Map}_c(M_0, Z)) \simeq \Omega\,\mathrm{Map}_c(M_0, Z)$$

as grouplike $E_1$-spaces. $\square$

# Part II
# The classification theorem

In this part of the talk we sketch a proof of Theorem 1.4 assuming that factorization homology satisfies $\otimes$-excision.

## 3 Factorization homology is symmetric monoidal

In this section we show that facotorization homology is symmetric monoidal (in the manifold) and explore some consequences.

**3.1 Lemma.** *Let $V$ be a symmetric monoidal $\infty$-category with colimits and assume that $\otimes$ preserves colimits seperately in each variable. Then for any $n$-disk algebra $A \in \mathbf{Alg}_n(V)$, the functor*

$$\int_{(-)} A \in \mathrm{Fun}(\mathbf{Mfld}_n, V)$$

*has a natural symmetric monoidal structure.*

*Proof sketch.* For any finite collection of $n$-manifolds $\{M_i\}_{i\in I}$, taking $I$-fold disjoint unions defines an equivalence of $\infty$-categories

$$\prod_{i\in I} \mathbf{Disk}_{n,/M_i} \xrightarrow{\sim} \mathbf{Disk}_{n,/(\coprod_{i\in I} M_i)} .$$

Since $A$ is an $n$-disk algebra, the square

$$\begin{tikzcd} \prod_{i\in I} \mathbf{Disk}_{n,/M_i} \arrow[r, "\sim"] \arrow[d, "A^{\times k}"'] & \mathbf{Disk}_{n,/(\coprod_{i\in I} M_i)} \arrow[d, "A"] \\ V^{\times k} \arrow[r, "\otimes"'] & V \end{tikzcd}$$

commutes. Hence

$$\int_{\coprod_{i\in I} M_i} A \simeq \operatorname{colim}\left( \prod_{i\in I} \mathbf{Disk}_{n,/M_i} \xrightarrow{A^{\times k}} V^{\times k} \xrightarrow{\otimes} V \right).$$

Since the symmetric monoidal structure on $V$ preserves colimtis separately in each variable, we see that

$$\begin{aligned} \int_{\coprod_{i\in I} M_i} A &\simeq \bigotimes_{i\in I} \operatorname{colim}\left( \mathbf{Disk}_{n,/M_i} \xrightarrow{A} V \right) \\ &= \bigotimes_{i\in I} \int_{M_i} A . \end{aligned}$$

□

**3.2 Remark.** For every $n$-manifold $M$, the $\infty$-category $\mathbf{Disk}_{n,/M}$ is actually sifted [HA, Proposition 5.5.2.15; 1, Corollary 3.22], so in Lemma 3.1 it is only necessary to assume that the tensor product commutes with sifted colimits separately in each variable.

**3.3 Corollary.** *Let $V$ be a symmetric monoidal $\infty$-category with colimits and assume that $\otimes$ preserves colimits seperately in each variable. Then factorization homology defines a functor*

$$\int : \mathbf{Alg}_n(V) \to \mathrm{Fun}^{\otimes}(\mathbf{Mfld}_n, V)\ .$$

**3.4 Observation.** Let $V$ be a symmetric monoidal $\infty$-category. Then precomposition with the fully faithful symmetric monoidal functor $i : \mathbf{Disk}_n \hookrightarrow \mathbf{Mfld}_n$ defines a functor

$$i^\star : \mathrm{Fun}^{\otimes}(\mathbf{Mfld}_n, V) \to \mathrm{Fun}^{\otimes}(\mathbf{Disk}_n, V) = \mathbf{Alg}_n(V)\ .$$

If $V$ is $\otimes$-presentable, then $i^\star$ admits a left adjoint

$$i_!^{\otimes} : \mathbf{Alg}_n(V) \to \mathrm{Fun}^{\otimes}(\mathbf{Mfld}_n, V)$$

given by *symmetric monoidal left Kan extension*: for $A \in \mathbf{Alg}_n(V)$, the symmetric monoidal functor $i_!(A)$ is the universal symmetric monoidal functor fitting into a diagram

$$\begin{tikzcd} \mathbf{Disk}_n \arrow[rr, "A"] \arrow[dr, hook, "i"'] & {} \arrow[d, phantom, "\Downarrow"] & V \\ & \mathbf{Mfld}_n \arrow[ur, dotted, "i_!^{\otimes}(A)"'] & \end{tikzcd} .$$

Since $i$ is fully faithful, the natural transformation $A \to i_!^{\otimes}(A) \circ i$ is an equivalence.

Since $V$ is presentable, the (ordinary) left Kan extension $i_!(A)$ of $A$ along $i$ has value at $M$ given by

$$\begin{aligned} i_!(A)(M) &\simeq \operatorname{colim}\left( \mathbf{Disk}_{n,/M} \longrightarrow \mathbf{Disk}_n \xrightarrow{A} V \right) \\ &\simeq \int_M A\ . \end{aligned}$$

We've already seen that the factorization homology $\int_{(-)} A$ of $A$ has a natural symmetric monoidal structure (Lemma 3.1) by checking things explicitly. One can use this to deduce that the ordinary left Kan extension $\int_{(-)} A$ agrees with the symmetric monoidal left Kan extension $i_!^{\otimes}(A)$. (See [1, Proposition 3.7].)

This shows that we have an adjunction

$$\int : \mathbf{Alg}_n(V) \rightleftarrows \mathrm{Fun}^{\otimes}(\mathbf{Mfld}_n, V) : i^\star\ .$$

Moreover, taking facotrization homology is fully faithful: if $U \in \mathbf{Disk}_n$, then since $i$ is fully faithful,

$$A(U) \xrightarrow{\sim} i^\star\left(\int_U A\right).$$

In summary, we've shown:

**3.5 Proposition.** *Let $V$ be $\otimes$-presentable symmetric monoidal $\infty$-category. Then we have an adjunction*

$$\int : \mathbf{Alg}_n(V) \rightleftarrows \mathrm{Fun}^{\otimes}(\mathbf{Mfld}_n, V) : i^\star .$$

*Moreover, the left adjoint $\int : \mathbf{Alg}_n(V) \to \mathrm{Fun}^{\otimes}(\mathbf{Mfld}_n, V)$ is fully faithful.*

# 4 Proof of the classification theorem

The next question is what is the essential image of $\int : \mathbf{Alg}_n(V) \hookrightarrow \mathrm{Fun}^{\otimes}(\mathbf{Mfld}_n, V)$? We'll take the following as given:

**4.1 Proposition** ([1, Lemma 3.18])**.** *Let $V$ be $\otimes$-presentable symmetric monoidal $\infty$-category. Then factorization satisfies $\otimes$-excision, hence defines a functor*

$$\int : \mathbf{Alg}_n(V) \hookrightarrow \mathbf{H}(\mathbf{Mfld}_n, V)$$

*landing in homology theories for $n$-manifolds.*

**4.2.** Assuming Proposition 4.1, we'll now prove Theorem 1.4. The idea of the proof is to use handelbody decompositions and collar gluings to build up from the case of disks.

*Proof of Theorem 1.4.* Fix a homology theory for $n$-manifolds $F \in \mathbf{H}(\mathbf{Mfld}_n, V)$. We want to show that for any $n$-manifold $M$, the counit

$$\int_M i^\star(F) \to F(M)$$

is an equivalence.

**Step 1** (disks)**.** Let $U \in \mathbf{Disk}_n$. Then since $\mathbf{Disk}_{n,/U}$ has a final object, by the definition of the restriction $i^\star(F)$ of $F$ to $\mathbf{Disk}_n$, the counit

$$\int_U i^\star(F) \to F(U)$$

is an equivalence.

**Step 2** (thickened spheres $S^k \times \mathbf{R}^{n-k}$)**.** Using induction we'll prove the claim when $M \cong S^k \times \mathbf{R}^{n-k}$ is a thickened sphere. The base case where $k = 0$ follows from Step 1.

For the inductive step, assume the result for $S^{k-1} \times \mathbf{R}^{n-k+1}$, and we'll prove the result for $S^k \times \mathbf{R}^{n-k}$. Choose the standard collar gluing $f : S^k \to [-1, 1]$ with $f^{-1}(0) = S^{k-1} \subset S^k$ the equator. By taking the product with $\mathbf{R}^{n-k}$, this gives rise to a collar gluing of $S^k \times \mathbf{R}^{n-k}$.

By our induction hypothesis and Step 1 we see that

$$\begin{aligned}
\int_{S^k\times\mathbf{R}^{n-k}} i^\star(F) &\simeq \Big(\int_{\mathbf{R}^k_{-1}\times\mathbf{R}^{n-k}} i^\star(F)\Big) \bigotimes_{\int_{S^{k-1}\times\mathbf{R}^{n-k+1}} i^\star(F)} \Big(\int_{\mathbf{R}^k_{+1}\times\mathbf{R}^{n-k}} i^\star(F)\Big) && (\otimes\text{-excision})\\
&\simeq F(\mathbf{R}^k_{-1}\times\mathbf{R}^{n-k}) \bigotimes_{F(S^{k-1}\times\mathbf{R}^{n-k+1})} F(\mathbf{R}^k_{+1}\times\mathbf{R}^{n-k}) && (\text{induction})\\
&\simeq F(S^k\times\mathbf{R}^{n-k}) && (\otimes\text{-excision})\,.
\end{aligned}$$

**Step 3** (use handelbody decompositions for $n \neq 4$)**.** We'll prove the claim for $n \neq 4$ by using the fact that when $n \neq 4$, all topological $n$-manifolds admit a handelbody decomposition. We prove the claim by inducting on the handel decomposition; the base case is Step 2.

For the inductive step, suppose that $M$ is obtained from $M'$ by attaching a handel of index $q$, so that there is a collar gluing

$$M \cong M' \cup^{S^q\times\mathbf{R}^{n-q}} \mathbf{R}^n\,,$$

where $\mathbf{R}^n$ is an open neighborhood of the $(q+1)$-handel in $M$. Since $F$ and $\int_{(-)} i^\star(F)$ agree on $M'$ (by induction), $S^q\times\mathbf{R}^{n-q}$ (by Step 2), and $\mathbf{R}^n$ (by Step 1) and both statisfy $\otimes$-excision, we deduce that

$$\int_M i^\star(F) \xrightarrow{\sim} F(M)\,.$$

This completes the proof when $n \neq 4$.

**Step 4** (topological 4-manifolds)**.** Now we treat the case of topological 4-manifolds. Without loss of generality we can assume that $M$ is connected. Then for any point $x \in M$, there exists a smooth structure on $M \smallsetminus \{x\}$, hence a handelbody decomposition of $M \smallsetminus \{x\}$. By the handelbody argument from Step 3, we deduce that

$$\int_{M\smallsetminus\{x\}} i^\star(F) \xrightarrow{\sim} F(M\smallsetminus\{x\})\,.$$

Applying $\otimes$-excision to the collar gluing

$$M \cong (M\smallsetminus\{x\}) \cup^{S^{n-1}\times\mathbf{R}} \mathbf{R}^n$$

proves the claim. □

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