HOW AND WHY TO USE FACTORIZATION HOMOLOGY

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"The reader uncomfortable with this language can substitute the words topological category for ∞-category wherever they occur in this paper to obtain the correct sense of the results, but they should then bear in mind the proviso that technical difficulties may then abound in making the statements literally true." Ayala-Francis

1. Part 1: Overview

Remark. A lot of what we'll talk about today works over a general stable, symmetric monoidal ∞-category, but we'll stick with $(\mathcal{C}\mathsf{h}_{\mathbb{Q}}, \oplus)$ or $(\mathcal{C}\mathsf{h}_{\mathbb{Q}}, \otimes)$.[1]

Let's start by discussing what *kind* of thing factorization homology is.

1.1. Basic Definitions.

Factorization homology takes in two inputs, an n-manifold M and something like an $\mathcal{E}_n$-algebra A valued in $\mathcal{C}\mathsf{h}_{\mathbb{Q}}$. It spits out "factorization homology of M with coefficients in A,"

$$\int_M A \in \mathcal{C}\mathsf{h}_{\mathbb{Q}}$$

Remark. Here $\mathcal{E}_n$ is the little n-disks operad. If we want to input to an actual $\mathcal{E}_n$-algebra, we need to require the manifold M to be framed. We will discuss a version of $\mathcal{E}_n$-algebras that works for manifolds without framing in the second part of this talk. For now, we'll assume our manifolds are all framed.

Example. Take $\mathcal{C}\mathsf{h}_{\mathbb{Q}}$ with tensor product. Let M be a framed n-manifold. Let X be a space. Take the free $\mathcal{E}_n$-algebra $\mathbb{F}_n(C_*X)$ on $C_*(X;\mathbb{Q}) \in \mathcal{C}\mathsf{h}_{\mathbb{Q}}$. Then

$$\mathbb{F}_n X \simeq C_*(\Omega^n \Sigma^n X; \mathbb{Q})$$

The factorization homology of M valued in a free $\mathcal{E}_n$-algebra is a labeled configuration space,

$$\int_M C_*\Omega^n\Sigma^n X \simeq \bigoplus_{i\geq 0} C_*(\mathsf{Conf}_i M) \bigotimes_{\Sigma_i} X^{\otimes i} = C_*(\mathsf{Conf}_X M; \mathbb{Q})$$

We will prove this in the second part of this talk.

In general, a good way to think about factorization homology is something that encodes points in M, each with a label in A so that when points collide, their labels multiply according to the algebra structure of A. A precursor to factorization homology along these lines is given in [17].

[1] Poincaré/Koszul duality itself requires additional conditions on the ∞-category.

1.2. **Manifold Invariant.** Why is it called factorization *homology*? First, note that it generalizes ordinary homology.

Example. Take $\mathcal{C}\mathsf{h}_{\mathbb{Q}}$ with symmetric monoidal structure given by $\oplus$. Take $V \in \mathcal{C}\mathsf{h}_{\mathbb{Q}}$ with its addition. Then

$$\int_M V = C_*(M; V)$$

We can also recover ordinary homology from an $\mathcal{E}_n$-algebra in $(\mathcal{C}\mathsf{h}_{\mathbb{Q}}, \otimes)$. Using the Dold-Kan construction, one can also show that

$$\int_M C_*\Omega^n K(\pi, i) \simeq C_{*+n-i}(M; \pi)$$

for $i \geq n$.

Factorization homology is a homology theory more suited for manifolds.

Theorem 1.1 (Ayala-Francis, [2])**.** *Factorization homology satisfies a version of the Eilenberg-Steenrod axioms more suited for manifolds.*

For example, factorization homology satisfies a version of excision for handle-body decompositions.

We have "seen" that factorization homology generalizes ordinary homology. This raises a natural question:

Question 1. Is this a pointless generalization?

Theorem 1.2 (Longoni-Salvatore, [15])**.** *No. Configuration spaces are not homotopy invariant.*

Thus $\mathsf{Conf}_X M$, and hence factorization homology over M, knows more about M than $H_*(M)$ could ever hope to.

1.3. **Motivation.** The core motivation for factorization homology is as a good manifold invariant. We list a few side motivations, some of which are secretly just to study manifolds again.

- Configuration spaces: Factorization homology is something like labeled configuration spaces, so it makes sense that information about configuration spaces would tell you something about factorization homology. Things work the other way as well. Using results about factorization homology, Knudsen in [12] gave a general, useful, and pretty description of the rational homology of unordered configuration spaces.
  - Configuration spaces are manifolds.
- Recasting old results in more natural settings: Bandklayder in [4] gave a new proof of the Dold-Thom theorem, using factorization homology, that is much slicker and intuitive.
- Knot theory: One can define a relative version of factorization homology that takes in an embedding $M \hookrightarrow N$ rather than a single manifold M. A natural collection of such embeddings to consider is knots $S^1 \hookrightarrow N$ in a manifold N. Various knot invariants can be defined in terms of factorization homology over the knot.
  - Knots are just manifolds in other manifolds.
- Hochschild homology: Factorization homology over S^1 takes in an $\mathcal{E}_1$-algebra; i.e., an associative algebra A. In this case,

$$\int_{S^1} A \simeq HH_\bullet(A)$$

  factorization homology over the circle is Hochschild homology. You might then consider factorization homology over higher dimensional spheres to be some sort of higher Hochschild homology.

- You might have thought that Hochschild homology is an algebraic thing, but the circle action is a huge part of it. The circle is a manifold.

- Physics:
  - • Costello's geometric construction of the Witten genus is in terms of factorization algebras/homology, [6].
  - • Costello-Gwilliam's work relating gauge theories to quantum groups uses factorization algebras as their intermediary, [7].
  - • TQFTS: Ayala-Francis have an almost complete proof of the Cobordism Hypothesis using factorization homology, [1].
  - - Physics is like studying the universe and stuff, and the universe itself is probably a manifold.

There's a not-so-subliminal message here: everything is manifolds.

1.4. **Goal of this Seminar.** The goal of this seminar is to become familiar with factorization homology: basic examples, construction, and techniques. We'll do this by learning about Ayala-Francis' result "Poincaré/Koszul duality." The rest of the first part of this talk will be motivating this result.

Let's start with our toy example: labeled configuration spaces. For this we follow the Benjamin Euclid Knudsen's notes, [13]. Recall that $C_*\mathsf{Conf}_X(-) = \int_{(-)} C_*\Omega^n\Sigma^n X$ defines a homology theory for n-manifolds.

Question 2. What is Poincaré duality for this homology theory?

First we need a candidate for cohomology.

Definition. Let X a space. Define a pointed space $\mathsf{Conf}_X(D^n, \partial D^n)$ as the quotient of $\mathsf{Conf}_X D^n$ by the relation $xu \simeq \emptyset$ for $x \in X$ and $u \in \partial D^n$.

Theorem 1.3 (McDuff)**.** *Let M be a framed n-manifold and X a connected space. There is an equivalence*

$$\mathsf{Conf}_X M \xrightarrow{\sim} \mathsf{Maps}_c(M, \mathsf{Conf}_X(D^n, \partial D^n))$$

The map in McDuff's theorem is called a scanning map, and her theorem is an example of an h-principle. One can think of the scanning map as follows. Take a labeled configuration $\bar{m} = \{x_\alpha m_\alpha\}$ for $x_\alpha \in X$ and $m_\alpha \in M$. The scanning map sends this configuration space to a compactly supported map $f_{\bar{m}} : M \to \mathsf{Conf}_X(D^n, \partial D^n)$. The map $f_{\bar{m}}$ takes in a point $u \in M$, "scans" a little disk $D^n_u \subset M$ centered at u for points of $\bar{m}$. Thus $f_{\bar{m}}(u)$ is the intersection of D^n_u with $\{x_\alpha m_\alpha\}$.

Remark. McDuff's theorem is also true for M not necessarily parallelizable. In this case, the statement is that of an equivalence

$$\mathsf{Conf}_X M \xrightarrow{\sim} \Gamma_c(M, \mathsf{Fr}_M \times_{O(n)} \mathsf{Conf}_X(D^n, \partial D^n)$$

where Fr_M is the frame bundle of M.

McDuff's theorem tells us what Poincaré duality should be for free $\mathcal{E}_n$-algebras,

$$\int_M \mathbb{F}_n V \simeq C_*\mathsf{Maps}_c(M, \mathsf{Conf}_X(D^n, \partial D^n))$$

We know that factorization homology generalizes ordinary homology in the case of $\Omega^n K(\pi, i)$ for $i \geq n$. Unfortunately, $K(\pi, i)$ is not an n-fold suspension so we can't apply McDuff's theorem. It would be nice to have a version of Poincaré duality that generalized McDuff's theorem as well as ordinary Poincaré duality.

We need a generalization of $\mathsf{Maps}_c(M, \mathsf{Conf}_X(D^n, \partial D^n))$. The first step is to see where $\Sigma^n X$ appears in this space.

Lemma 1.4. *There is an equivalence*

$$\mathsf{Conf}_X(D^n, \partial D^n) \simeq \Sigma^n X$$

For a proof, see [13, Prop. 6.1.12].

Thus for M framed, using our claimed computation of factorization homology of a free algebra, we have a splitting result

$$\bigoplus_{i\geq 0} C_*(\mathsf{Conf}_i M \times_{\Sigma_i} X^{\wedge i}) \simeq \int_M C_*\Omega^n\Sigma^n X \simeq C_*\mathsf{Maps}_c(M, \Sigma^n X)$$

Remark. This splitting was obtained from McDuff's theorem. New proofs of similar splitting results for mapping spaces are given by Bandklayder in [5].

Let's put Poincaré duality in this form:

$$\begin{array}{ccc} H_{*+n-i}(M;\mathbb{Q}) & \xrightarrow{\cong} & H_c^{i-*}(M;\mathbb{Q}) \\ \downarrow \cong & & \downarrow \cong \\ H_* \int_M C_*\Omega^n K(\pi, i) & \dashrightarrow & \pi_*\mathsf{Maps}_c(M, K(\pi, i)) \end{array}$$

This leads to the following guess

$$\int_M C_*\Omega^n Z \xrightarrow{\sim} C_*\mathsf{Maps}_c(M, Z)$$

There is always such a map, but this is only an equivalence under certain connectivity assumptions. The first step in this seminar will be to cover the proof Poincaré duality in this setting, which is due to Segal, Salvator, and Lurie.

Theorem 1.5 (Non-abelian Poincaré Duality (NAPD))**.** *Let M a framed n-manifold and Z an $(n-1)$-connected space. Then there is an equivalence*

$$\int_M C_*\Omega^n Z \xrightarrow{\sim} C_*\mathsf{Maps}_c(M, Z)$$

To recap, non-abelian Poincaré Duality reduces to McDuff's theorem in the case $Z = \Sigma^n X$ and reduces to Poincaré duality in the case $Z = K(\pi, i)$.

Remark. Like McDuff's theorem, NAPD is also true for M not necessarily framed and $\mathcal{E}_n$-algebras replaced by a non-framed version.

Remark. Gaitsgory and Lurie use non-abelian Poincaré duality in their work on the Weil's conjecture on Tamagawa numbers, [10].

Proof Idea. By Theorem

Let Z be a connected space. Then C^*Z can be given an $\mathcal{E}_\infty$-algebra structure from the diagonal maps on Z. In the following theorem, we regard C^*Z as an $\mathcal{E}_n$-algebra.

Theorem 1.6. *Let Z be a connected space. Then, for M a framed n-manifold, there is an equivalence*

$$C^*\mathsf{Maps}_c(M, Z) \simeq \int_M C^*Z$$

For a proof see [2, Prop. 5.1] or Kumar's notes [14].

If Z is $(n-1)$-connected, we can combine this theorem with Non-abelian Poincaré Duality, to get

$$\left(\int_M C_*\Omega^n Z\right)^\vee \simeq \int_M C^* Z$$

You should like this since it makes it feel like you can move the duality to the inside, where it's something more algebraic.

Question 3. Given an $\mathcal{E}_n$-algebra A, does there exist a "dual" $\mathcal{E}_n$-algebra $??_A$ so that

$$\left(\int_M A\right)^\vee \simeq \int_M ??_A$$

Idea 1. Cheat. We want $??_A$ to be the dual of A. What's a good name for the dual of A? Well $\mathbb{D}(A)$ is good. It depends on n, so maybe $\mathbb{D}^n(A)$ is better. But $\mathbb{D}^n$ already means something, the n-disk. Wouldn't it be great if $\mathbb{D}^n(A) = \int_{\mathbb{D}^n} A$?

Idea 2 In NAPD, the dual of $\Omega^n Z$ was Z, an n-fold delooping. If M is compact and contractible, like $\mathbb{D}^n$ is, then NAPD gives

$$\int_M C_*\Omega^n Z \simeq C_*\mathsf{Maps}_c(M, Z) \simeq Z$$

Pretending for a second that we have defined factorization homology for manifolds with boundary (or defined it at all), this says that factorization homology over the n-disk is an n-fold delooping. So our cheating guess seems right.

Idea 3. If everything was nice, $\mathbb{D}$ would define a duality on $\mathcal{E}_n$-algebras. The words "duality" and "$\mathcal{E}_n$-algebras" might remind you of the fact that the operad $\mathcal{E}_n$ is self Koszul dual, up to a shift, [9], [11].[2] Perhaps one can use this to define a Koszul duality on (augmented) $\mathcal{E}_n$-algebras, and thereby obtain a guess for $\mathbb{D}(-)$.

Remark. For the first idea to work, we would need to introduce factorization homology over manifolds with boundary. For the third idea to work, we would need to introduce factorization homology with coefficients in augmented $\mathcal{E}_n$-algebras. We'll address these issues in the second part of this talk.

The goal of this seminar is to prove the following lovely theorem of Ayala-Francis, [3].

Theorem 1.7 (Poincaré/Koszul Duality)**.** *Let M be a connected, compact n-manifold. Let A be an augmented n-disk algebra with values in $\mathsf{Ch}_{\mathbb{Q}}$. If the kernel $\ker(A \to \mathbb{Q})$ is connected, and $H_i A$ is finite-dimensional for all i, then there is an equivalence*

$$\left(\int_M A\right)^\vee \simeq \int_M \mathbb{D}^n(A)$$

where the underlying chain complex of $\mathbb{D}^n(A)$ is $\int_{\mathbb{D}^n} A$.

[2] This is proven in the case of chain complexes by Fresse and at the level fo homology by Getzler and Jones.

2. Part 2: More Details and Examples

2.1. Disk Categories.

Definition. Let $\mathcal{M}\mathsf{fld}_n$ be the ∞-category of n-manifolds. This has objects n-manifolds and morphisms smooth embeddings.

Let $\mathcal{D}\mathsf{isk}_n \subset \mathcal{M}\mathsf{fld}_n$ be the full ∞-subcategory consisting of manifolds isomorphic to finite disjoint unions of Euclidean spaces.

Remark. If you are pretending these are just topological categories, the mapping spaces $\mathsf{Emb}(M, N)$ are given the compact-open topology.

Remark. We consider the empty set to be an object of $\mathcal{M}\mathsf{fld}_n$ for every n.

Remark. There are several other versions of disk categories (and categories of manifolds) that we will consider during this seminar.

- Framed disks: $\mathcal{D}\mathsf{isk}_n^{\mathrm{fr}}$ will have the same objects of $\mathcal{D}\mathsf{isk}_n$ but with framed embeddings as morphisms. A framed embedding is an embedding $M \to N$ so that the given framing on M and the pulled back framing commute up to a chosen homotopy.
- Pointed disks: $\mathcal{D}\mathsf{isk}_n^+$ will have objects $\bigvee_I(\mathbb{R}^n_+)$ and morphisms embeddings away from the preimage of the point $+$.
- Boundaries: $\mathcal{D}\mathsf{isk}_n^\partial$ will have objects $\coprod_I \mathbb{R}^n \sqcup \coprod_J \mathbb{R}^n_{\geq 0}$. Here $\mathbb{R}^n_{\geq 0} = \mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$, and not $(\mathbb{R}_{\geq 0})^{\times n}$.

Note that $\mathcal{D}\mathsf{isk}_n$ is a symmetric monoidal ∞-category with tensor product given by disjoint union. Given an n-manifold M, let $\mathcal{D}\mathsf{isk}_{n/M}$ denote the over category. Objects of $\mathcal{D}\mathsf{isk}_{n/M}$ are embeddings $U \hookrightarrow M$ so that U is isomorphic to a finite disjoint union $\sqcup \mathbb{R}^n$ of Euclidean spaces. Morphisms in this category are triangles

$$\begin{array}{ccc} U & \longrightarrow & M \\ \downarrow & \nearrow & \\ V & & \end{array}$$

that commute up to a chosen isotopy. The over category $\mathcal{D}\mathsf{isk}_{n/M}$ comes with a forgetful functor

$$\mathcal{D}\mathsf{isk}_{n/M} \to \mathcal{D}\mathsf{isk}_n$$

Definition. An n-disk algebra A with values in a symmetric monoidal ∞-category $\mathcal{V}$ is a symmetric monoidal functor

$$A : \mathcal{D}\mathsf{isk}_n \to \mathcal{V}$$

Let $\mathsf{Alg}_n(\mathcal{V})$ denote the category of n-disk algebras.

An augmented n-disk algebra is a symmetric monoidal functor

$$\mathcal{D}\mathsf{isk}_n^+ \to \mathcal{V}$$

2.2. $\mathcal{E}_n$-algebras.

Relationship between $\mathcal{E}_n$-algebras and n-disk algebras. If we redo everything above with framed manifolds, framed embeddings, and such, then a *framed* n-disk algebra is the same as an $\mathcal{E}_n$-algebra. The equivalence goes as follows. Given a framed n-disk algebra A with values in $\mathcal{V}$, define an $\mathcal{E}_n$-algebra in $\mathcal{V}$ by $A(\mathbb{R}^n)$ and action

$$\mathsf{Emb}(\coprod_I \mathbb{R}^n, \mathbb{R}^n) \otimes A(\mathbb{R}^n)^{\otimes I} \to A(\mathbb{R}^n)$$

by identifying $A(\mathbb{R}^n)^{\otimes I} \simeq A(\coprod_I \mathbb{R}^n)$ and applying the given embedding.

More precisely, there is an equivalence of categories

$$\mathsf{Alg}_{\mathcal{D}\mathsf{isk}_n^{\mathrm{fr}}}(\mathcal{V}) \cong \mathsf{Alg}_{\mathcal{E}_n}(\mathcal{V})$$

For a proof see [16].

Definition. Let M be an n-manifold and A an n-disk algebra valued in $\mathcal{V}$. The *factorization homology of M with coefficients in A* is the homotopy colimit

$$\mathsf{colim}\left(\mathcal{D}\mathsf{isk}_{n/M} \to \mathcal{D}\mathsf{isk}_n \xrightarrow{A} \mathcal{V}\right)$$

2.3. **Homology Theories for Manifolds.** Factorization homology satisfy a version, more suited to manifolds, of the Eilenberg-Steenrod axioms for homology theories. The main axiom of such theories is called "$\otimes$-excision."

Definition. A symmetric monoidal functor

$$F : \mathsf{Mfld}_n \to \mathcal{C}\mathsf{h}_{\mathbb{Q}}$$

satisfies $\otimes$-exision if, for every collar-gluing $U \bigcup_{V\times\mathbb{R}} U' \simeq W$, the canonical morphism

$$F(U) \bigotimes_{F(V\times\mathbb{R})} F(U') \to F(W)$$

is an equivalence.

Here $F(V \times \mathbb{R})$ inherits an $\mathcal{E}_1$-algebra structure from the copy of $\mathbb{R}^1$,

$$\mathsf{Emb}^{\mathrm{fr}}(\coprod_I \mathbb{R}, \mathbb{R}) \otimes F(V \times \mathbb{R})^{\otimes I} \simeq \mathsf{Emb}^{\mathrm{fr}}(\coprod_I \mathbb{R}, \mathbb{R}) \otimes F(V \times (\coprod_I \mathbb{R})) \to F(V \times \mathbb{R})$$

The tensor product

$$F(U) \bigotimes_{F(V\times\mathbb{R})} F(U') \to F(W)$$

is then the tensor product in modules over the $\mathcal{E}_1$-algebra $F(V \times \mathbb{R})$. One reason we have restricted to collar-gluings is so that this tensor product makes sense.

Definition. The ∞-category of *homology theories for n-manifolds* valued in $\mathcal{C}\mathsf{h}_{\mathbb{Q}}$ is the full ∞-subcategory

$$\mathbb{H}(\mathcal{M}\mathsf{fld}_n, \mathcal{C}\mathsf{h}_{\mathbb{Q}}) \subset \mathsf{Fun}^{\otimes}(\mathcal{M}\mathsf{fld}_n, \mathcal{C}\mathsf{h}_{\mathbb{Q}})$$

of symmetric monoidal functors that satisfy $\otimes$-excision.

Not only is factorization homology a homology theory for n-manifolds, it also is the only such thing.

Theorem 2.1 (Ayala-Francis, [2])**.** *There is an equivalence*

$$\int : \mathsf{Alg}_n(\mathcal{C}\mathsf{h}_{\mathbb{Q}}) \leftrightarrows \mathbb{H}(\mathcal{M}\mathsf{fld}_n, \mathcal{C}\mathsf{h}_{\mathbb{Q}}) : \mathrm{ev}_{\mathbb{R}^n}$$

Remark. One can replace $\mathcal{C}\mathsf{h}_{\mathbb{Q}}$ with a general symmetric monoidal ∞-category $\mathcal{V}$ as long as $\mathcal{V}$ is "$\otimes$-presentable. For details, see [2].

2.4. **Examples.** We compute factorization homology $\int_M A$ for simple choices of M and A.

Example. Take $M = \mathbb{R}^n$. Then $\mathcal{D}\mathsf{isk}_{n/\mathbb{R}^n}$ has a final object given by the identity map $\mathbb{R}^n = \mathbb{R}^n$. Thus the colimit is given by evaluation on $\mathbb{R}^n$,

$$\int_{\mathbb{R}^n} A = \mathsf{colim}\left(\mathcal{D}\mathsf{isk}_{n/\mathbb{R}^n} \to \mathcal{D}\mathsf{isk}_n \xrightarrow{A} \mathcal{V}\right) = A(\mathbb{R}^n)$$

Example. Take $M = \coprod_I \mathbb{R}^n$. Then $\mathcal{D}\mathsf{isk}_{n/M}$ again has a final object and as above we obtain

$$\int_{\coprod_I \mathbb{R}^n} A \simeq A(\coprod_I \mathbb{R}^n) \cong A(\mathbb{R}^n)^{\otimes I}$$

Here we are seeing the fact that

$$\int_{(-)} A : \mathcal{M}\mathsf{fld}_n \to \mathcal{C}\mathsf{h}_{\mathbb{Q}}$$

is a symmetric monoidal functor.

Example. Take $M = S^1$, as a framed manifold. Note that an $\mathcal{E}_1$-algebra A is the same as an associative algebra $\bar{A} := A(\mathbb{R}^1)$. We will use excision to compute $\int_{S^1} A$. Express S^1 as a collar-gluing

$$S^1 \cong \mathbb{R} \cup_{S^0 \times \mathbb{R}} \mathbb{R}$$

By $\otimes$-excision, we have

$$\int_{S^1} A \simeq \left(\int_{\mathbb{R}} A\right) \bigotimes_{(\int_{S^0\times\mathbb{R}} A)} \left(\int_{\mathbb{R}} A\right) \simeq \bar{A} \bigotimes_{\bar{A}\otimes\bar{A}^{\mathrm{op}}} \bar{A} = HH_\bullet(A)$$

where we obtained $\bar{A} \otimes \bar{A}^{\mathrm{op}}$ because the two copies of $\mathbb{R}^1$ in $S^0 \times \mathbb{R}^1 \subset S^1$ are oriented differently.

We have a functor $\mathsf{Alg}_n(\mathcal{V}) \to \mathcal{V}$ given by evaluating on $\mathbb{R}^n$. This is the "forgetful functor."

Definition. The left adjoint to the forgetful functor is the free functor

$$\mathbb{F}_n \colon \mathcal{V} \to \mathsf{Alg}_n(\mathcal{V})$$

Example. Consider the free n-disk algebra on $V \in \mathcal{C}\mathsf{h}_{\mathbb{Q}}$. This sends a disjoint union $\coprod_k \mathbb{R}^n$ to

$$\bigoplus_{0\leq i} C_*(\mathsf{Emb}(\coprod_i \mathbb{R}^n, \coprod_k \mathbb{R}^n)) \bigotimes_{\Sigma_i} V^{\otimes i}$$

Example. We can similarly define a free framed n-disk algebra. In the framed case,

$$\mathbb{F}_n V(\mathbb{R}^n) = \bigoplus_{i\geq 0} C_* \mathsf{Emb}^{\mathrm{fr}}(\coprod_i \mathbb{R}^n, \mathbb{R}^n) \bigotimes_{\Sigma_i} V^{\otimes i}$$

Since $\mathsf{Emb}^{\mathrm{fr}}(\coprod_i \mathbb{R}^n, \mathbb{R}^n) \simeq \mathsf{Conf}_i(\mathbb{R}^n)$, this agrees with the free $\mathcal{E}_n$-algebra on V.

Proposition 2.2. *For M a framed manifold, and $V \in \mathcal{C}\mathsf{h}_{\mathbb{Q}}$, we have*

$$\int_M \mathbb{F}_n V \simeq \bigoplus_{0\leq i} C_*(\mathsf{Conf}_i M) \bigotimes_{\Sigma_i} V^{\otimes i}$$

A similar statement is true in the non-framed case, we're just being lazy.

For $U \cong \coprod_I \mathbb{R}^n$ we have

$$\mathbb{F}_n V(U) = \left(\bigoplus_{i \geq 0} C_*(\mathsf{Conf}_i \mathbb{R}^n) \bigotimes_{\Sigma_i} V^{\otimes i} \right)^{\otimes I} \cong \bigoplus_{i \geq 0} C_*(\mathsf{Conf}_i U) \bigotimes_{\Sigma_i} V^{\otimes i}$$

Thus

$$\begin{aligned} \int_M \mathbb{F}_n V &= \underset{U \in \mathcal{D}\mathsf{isk}_{n/M}}{\mathsf{colimit}} \bigoplus_{i \geq 0} \left(C_*(\mathsf{Conf}_i U) \bigotimes_{\Sigma_i} V^{\otimes i} \right) \\ &= \bigoplus_{i \geq 0} \underset{U \in \mathcal{D}\mathsf{isk}_{n/M}}{\mathsf{colimit}} \left(C_*(\mathsf{Conf}_i U) \bigotimes_{\Sigma_i} V^{\otimes i} \right) \end{aligned}$$

Let $\mathsf{Disk}^{\mathrm{fr}}_{n/M}$ denote the *ordinary* category of framed n-disks in M. We'll just show things for the ordinary category $\mathsf{Disk}^{\mathrm{fr}}_{n/M}$, instead of for the ∞-category $\mathcal{D}\mathsf{isk}^{\mathrm{fr}}_{n/M}$. It turns out that this is sufficient:

Theorem 2.3 ([2])**.** *The functor $\mathsf{Disk}^{\mathrm{fr}}_{n/M} \to \mathcal{D}\mathsf{isk}^{\mathrm{fr}}_{n/M}$ is a localization. Hence factorization homology can be computed as a colimit over $\mathsf{Disk}^{\mathrm{fr}}_{n/M}$.*

For a more direct proof in the ∞-category case, see Proposition 5.5.2.13 of [16].

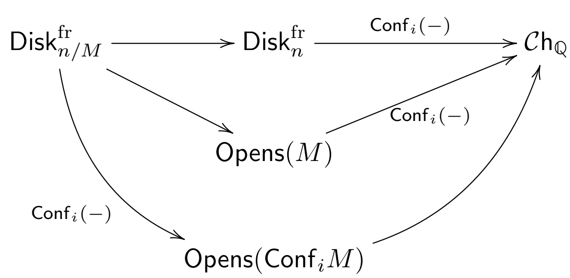
To compute this colimit, we use a hypercover argument. This is theorem A.3.1 in [16]. Also see [8] and [13].

Theorem 2.4 (Seifert-van Kampen Theorem)**.** *Let X be a topological space. Let $\mathsf{Opens}(X)$ denote the poset of open subsets of X. Let $\mathcal{C}$ be a small category and let $F : \mathcal{C} \to \mathsf{Opens}(X)$ be a functor. For every $x \in X$, let $\mathcal{C}_x$ denote the full subcategory of $\mathcal{C}$ spanned by those objects $C \in \mathcal{C}$ such that $x \in F(C)$. If for every $x \in X$, the simplicial set $N(\mathcal{C}_x)$ is weakly contractible, then the canonical map*

$$\underset{C \in \mathcal{C}}{\mathsf{colim}} \mathsf{Sing}(F(C)) \to \mathsf{Sing}(X)$$

exhibits the simplicial set $\mathsf{Sing}(X)$ as a homotopy colimit of the diagram $\{\mathsf{Sing}(F(C))\}_{C \in \mathcal{C}}$.

To use the Seifert-van Kampen theorem, consider the following commutative diagram,



Let $\bar{x} = (x_1, \ldots, x_i) \in \mathsf{Conf}_i M$. The category $(\mathsf{Disk}^{\mathrm{fr}}_{n/M})_{\bar{x}}$ contains embed disks $U \hookrightarrow M$ so that $\{x_1, \ldots, x_i\}$ is in U. By the Seifert-van Kampen theorem, if $B(\mathsf{Disk}^{\mathrm{fr}}_{n/M})_{\bar{x}} \simeq *$, then

$$\underset{U \in \mathsf{Disk}_{n/M}}{\mathsf{colimit}} \ \mathsf{Conf}_i U \simeq \mathsf{Conf}_i M$$

To show that this category is contractible, we will show it is cofiltered.

Definition. A nonempty *ordinary* category $\mathcal{C}$ is cofiltered if

1) for every pair $U, V \in \mathcal{C}$ there exists $W \in \mathcal{C}$ and maps $W \to U$ and $W \to V$, and
2) given two maps $u, v : X \to Y$ in $\mathcal{C}$, there exists $Z \in \mathcal{C}$ and a map $w : Z \to X$ so that $uw = vw$.

Computation in the free case. Let $U, V \in (\mathsf{Disk}^{\mathrm{fr}}_{n/M})_{\bar{x}}$. We need to find a finite disjoint union of euclidean spaces $W \to M$ containing $(x_1, \ldots, x_i)$ and maps $W \to U$ and $W \to V$. Note that $U \cap V$ contains $\bar{x}$, but may not be a disjoint union of euclidean spaces. However, we can find a small disk around each x_i and still in $U \cap V$. The second condition is satisfied since $\mathsf{Disk}^{\mathrm{fr}}_{n/M}$ is a poset.

Thus $(\mathsf{Disk}^{\mathrm{fr}}_{n/M})_{\bar{x}}$ cofiltered, and hence contractible. Applying the Seifert-van Kampen theorem, (and adding in a few details about V) we get

$$\int_M \mathbb{F}_n V \simeq \bigoplus_{i \geq 0} C_*(\mathsf{Conf}_i M) \otimes_{\Sigma_i} V^{\otimes i}$$

□

Definition. Let $V \in \mathcal{C}\mathsf{h}_{\mathbb{Q}}$. The *trivial augmented n-disk algebra* on V, denoted $t^{\mathrm{aug}}_{\mathrm{Alg}} V$ takes value $V \oplus \mathbb{1}$ on $\mathbb{R}^n_+$ and multiplication maps that factor through the augmentation,

$$\begin{array}{ccc} (V \oplus \mathbb{1})^{\otimes 2} & \xrightarrow{\qquad\qquad} & V \oplus \mathbb{1} \\ & \searrow \quad \nearrow & \\ & \mathbb{1} & \end{array}$$

Recall that $\mathbb{D}^n(A)$ is supposed to be a sort of Koszul duality. Now the Koszul dual of a free algebra is trivial. Using Poincaré/Koszul duality, we can get at $\int_M t^{\mathrm{aug}}_{\mathrm{Alg}} V$.

References

[1] Ayala, David and Francis, John. *The cobordism hypothesis.*
[2] Ayala, David and Francis, John. *Factorization Homology of Topological Manifolds*
[3] Ayala, David and Francis, John. *Poincaré/Koszul Duality*
[4] Bandklayder, Lauren. *A proof of the Dold-Thom theorem via factorization homology.*
[5] Bandklayder, Lauren. *Stable splitting of mapping spaces via non-abelian Poincaré duality.*
[6] Costello, Kevin. *A geometric construction of the Witten genus I,II.*
[7] Costello, K. and Gwilliam, O. *Factorization algebras in quantum field theory Vol 1,2.*
[8] Dugger, D. and Isaksen, D. *Topological hypercovers and $\mathbb{A}^1$-realizations.*
[9] Fresse, Benoit. *Koszul duality of* E_n*-operads.*
[10] Gaitsgory, Dennis and Lurie, Jacob. *Weil's conjecture for function fields.*
[11] Getzler, Ezra. and Jones, John. *Operads, homotopy algebra and iterated integrals for double loop spaces.*
[12] Knudsen, Ben. *Betti numbers and stability for configuration spaces via factorization homology.*
[13] Knudsen, Ben. *Configuration spaces in algebraic topology.*
[14] Kumar, Nilay. *Factorization homology.* Available at `https://sites.math.northwestern.edu/~nilay/pdf/factorization_homology.pdf`
[15] Longoni, R. and Salvatore, R. *Configuration spaces are not homotopy invariant.*
[16] Lurie, Jacob. *Higher Algebra.*
[17] Salvatore, R. *Configuration spaces with summable labels.*