Cofinality & categories of disks

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Abstract

In this note we recall some facts cofinality and a few cofinality results that show that it is possible to compute factorization homology over three different categories of disks.

Contents

1 Cofinality & localiations

1.1 Definition. Let C and D be ∞-categories. We say that a functor $F\colon C \to D$ is:

(1.1.1) ***colimit-cofinal*** if for every right fibration $X \to D$, precomposition with F induces an equivalence

$$\mathrm{Map}_{/D}(D, X) \to \mathrm{Map}_{/D}(C, X)\,.$$

(1.1.2) ***limit-cofinal*** if for every left fibration $X \to D$, precomposition with F induces an equivalence

$$\mathrm{Map}_{/D}(D, X) \to \mathrm{Map}_{/D}(C, X)\,.$$

1.2 Lemma. *Let C and D be ∞-categories. If a functor $F\colon C \to D$ is colimit-cofinal or limit-cofinal, then F is a weak homotopy equivalence.*

1.3 Theorem ([HTT, Proposition 4.1.1.8 & Theorem 4.1.3.1])**.** *Let C and D be ∞-categories. The following are equivalent for a functor $F\colon C \to D$:*

(1.3.1) *The functor F is colimit-cofinal.*

(1.3.2) **Quillen's Theorem A**: *For every object $d \in D$, the ∞-category $C \times_D D_{d/}$ is weakly contractible.*

(1.3.3) *For every* ∞*-category* E *and functor* $G\colon D \to E$*, the colimit* $\operatorname{colim}_D G$ *exists if and only if the colimit* $\operatorname{colim}_C GF$ *exists, and when both colimits exist the natural morphism*

$$\operatorname{colim}_D G \to \operatorname{colim}_C GF$$

is an equivalence in E*.*

1.4 Definition. Let C be an ∞-category and $W \subset \operatorname{Mor}(C)$ a class of morphisms in C. The ***localization*** of C at W is an ∞-category $C[W^{-1}]$ equipped with a functor $L\colon C \to C[W^{-1}]$ satisfying the following universal property:

(1.4.1) The functor L sends all morphisms in W to equivalences in $C[W^{-1}]$.

(1.4.2) For any ∞-category D, precomposition with L defines a fully faithful functor

$$L^{\star}\colon \operatorname{Fun}(C[W^{-1}], D) \hookrightarrow \operatorname{Fun}(C, D)$$

with essential image those functors $C \to D$ that send every morphism in W to an equivalence in D.

1.5 Remark. See [2, §7.1] for an excellent discussion of localizations of ∞-categories. In particular, [2, Proposition 7.1.3] shows that localizations always exist.

1.6 Lemma. *Let* C *be an* ∞*-category and* $W \subset \operatorname{Mor}(C)$ *a class of morphisms in* C*. Then the localization functor* $C \to C[W^{-1}]$ *is both limit-cofinal and colimit-cofinal.*

Proof. Recall that a functor $p\colon X \to Y$ is conservative if a morphsim f in X is an equivalence if and only if $p(f)$ is an equivalence in Y. Since both left and right fibrations are conservative functors [2, Proposition 3.4.8], it suffices to prove that for any conservative functor $p\colon X \to C[W^{-1}]$, the induced map

$$\operatorname{Map}_{/C[W^{-1}]}(C[W^{-1}], X) \to \operatorname{Map}_{/C[W^{-1}]}(C, W)$$

is an equivalence. Since $p\colon X \to C[W^{-1}]$ is a conservative functor, any functor $C \to X$ over $C[W^{-1}]$ necessarily carries all morphisms in W to equivalences in X. The unviersal property of the localization $C[W^{-1}]$ implies that any functor $C \to X$ over $C[W^{-1}]$ factors essentially uniquely through $C[W^{-1}]$, that is to say,

$$\operatorname{Map}_{/C[W^{-1}]}(C[W^{-1}], X) \xrightarrow{\sim} \operatorname{Map}_{/C[W^{-1}]}(C, W)\,. \qquad \square$$

1.7 Remark. See [2, Proposition 7.1.10] for an alternative proof of Lemma 1.6.

2 Categories of disks

2.1 Notation. For a nonnegative integer n, write Mfld_n for the 1-category of n-manifolds and embeddings. Write $\mathrm{Disk}_n \subset \mathrm{Mfld}_n$ for the full subcategory spanned by those n-manifolds isomorphic to a finite disjoint union of Euclidean spaces. Both Disk_n and Mfld_n have symmetric monoidal structures given by disjoint union.

We write $\mathbf{Mfld}_n$ for the ∞-category associated to the topological category[1] with objects n-manifolds and morphism spaces the spaces $\mathrm{Emb}(M,N)$ of embeddings $M \hookrightarrow N$ with the compact-open topology. We write $\mathbf{Disk}_n \subset \mathbf{Mfld}_n$ for the full subcategory spanned by those n-manfiolds isomorphic toa finite disjoint union of Euclidean spaces.

Ayala and Francis show that the ∞-category $\mathbf{Disk}_{n,/M}$ can be obtained from the 1-category $\mathrm{Disk}_{n,/M}$ by inverting those morphisms that are isotopic to an isomorphism. This is a somewhat technical result.

2.2 Theorem ([1, Proposition 2.19]). *Let M be a manifold and write I_M for the collection of morphisms in* $\mathrm{Disk}_{n,/M}$ *that are isotopic to an isomorphism. Then the functor*

$$\mathrm{Disk}_{n,/M} \to \mathbf{Disk}_{n,/M}$$

exhibits $\mathbf{Disk}_{n,/M}$ *as the localization* $\mathrm{Disk}_{n,/M}[I_M^{-1}]$ *of* $\mathrm{Disk}_{n,/M}$ *at* I_M.

In particular, the functor $\mathrm{Disk}_{n,/M} \to \mathbf{Disk}_{n,/M}$ *is colimit-cofinal.*

2.3 Notation. Let M be an n-manifold. Write $\mathrm{Disj}(M) \subset \mathrm{Open}(M)$ for the full subposet spanned by those open sets $U \subset M$ that are homeomorphic to a finite disjoint union of Euclidean spaces $\mathbf{R}^n$.

2.4 Observation. Let M be an n-manifold. Choose a parametrization of each open disk in M. This choice of parameterization gives rise to a functor $\gamma\colon \mathrm{Disj}(M) \to \mathrm{Disk}_{n,/M}$. We also write γ for the composite $\gamma\colon \mathrm{Disj}(M) \to \mathbf{Disk}_{n,/M}$.

Essentially the same argument that Minta presented last talk using Lurie's Seifert–van Kampen Theorem [HA, Theorem A.3.1] proves the following:

2.5 Proposition ([HA, Proposition 5.5.2.13]). *Let M be an n-manoifold and choose a parameterization of each open disk in M. Then the functor $\gamma\colon \mathrm{Disj}(M) \to \mathbf{Disk}_{n,/M}$ of Observation 2.4 is colimit-cofinal.*

2.6. Let M be an n-manifold. Theorem 2.2 and Proposition 2.5 show that we can compute the factorization homology of M valued in an n-disk algebra $A\colon \mathbf{Disk}_n \to V$ with values in a symmetric monoidal ∞-category V with colimits in three ways:

(2.6.1) By definition, as the colimit of the diagram

$$\mathbf{Disk}_{n,/M} \longrightarrow \mathbf{Disk}_n \xrightarrow{A} V\,.$$

(2.6.2) As the colimit of the diagram

$$\mathrm{Disk}_{n,/M} \longrightarrow \mathbf{Disk}_{n,/M} \longrightarrow \mathbf{Disk}_n \xrightarrow{A} V$$

indexed by the 1-category $\mathrm{Disk}_{n,/M}$.

[1] As a quasicategory, is the simplicial nerve of the fibrant simplicial category obtained from the topological category by applying the singular simplicial set functor $\mathrm{Sing}\colon \mathbf{Top} \to s\mathbf{Set}$.

(2.6.3) After choosing a parametrization of every disk in M, as the colimit of the diagram

$$\mathrm{Disj}(M) \xrightarrow{\gamma} \mathbf{Disk}_{n,/M} \longrightarrow \mathbf{Disk}_n \xrightarrow{A} V$$

indexed by the poset $\mathrm{Disj}(M)$.

2.7 Question. So what is the advantage of the ∞-category $\mathbf{Disk}_{n,/M}$ over the 1-category $\mathrm{Disk}_{n,/M}$, or the poset $\mathrm{Disj}(M)$ if we can compute factorization homology in any of these three ways?

2.8. First of all, the functor $\gamma\colon \mathrm{Disj}(M) \to \mathbf{Disk}_{n,/M}$ depends on a choice of parametrization of every disk in M, which is unnatural and cannot be made functorially. Second, the poset $\mathrm{Disj}(M)$ generally doesn't have any nice properties; $\mathrm{Disj}(M)$ generally is not filtered[2]. The 1-category $\mathrm{Disk}_{n,/M}$ generally isn't filtered either. The important feature of $\mathbf{Disk}_{n,/M}$ is that it is *sifted* (whereas $\mathrm{Disk}_{n,/M}$ generally is not).

2.9 Definition. An ∞-category C is ***sifted*** if C is nonempty and the diagonal functor $C \to C \times C$ is colimit-cofinal.

2.10 Examples.

(2.10.1) Filtered ∞-categories are sifted [HTT, Example 5.5.8.3].

(2.10.2) The 1-category $\mathbf{\Delta}^{op}$ is sifted [HTT, Lemma 5.5.8.4].

(2.10.3) If C is a sifted ∞-category, then C is weakly contractible [HTT, Proposition 5.5.8.7].

2.11 Proposition ([HA, Proposition 5.5.2.15]). *For every n-manifold M, the ∞-category $\mathbf{Disk}_{n,/M}$ is sifted.*

References

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2. D.-C. Cisinski, *Higher categories and homotopical algebra*, Preprint available at www.mathematik.uni-regensburg.de/cisinski/CatLR.pdf, May 2018.

[2] Although $\mathrm{Disj}(M)^{op}$ *is* filtered, but that's not helpful for this purpose since we want to take a colimit over $\mathrm{Disj}(M)$.