MIT 18.675 FALL 2019. LECTURE TOPICS

The notes below give a summary of the main topics covered in each lecture, and relevant reading. For lectures that closely follow our primary textbook [LG16], the notes will be very brief. This document will be updated throughout the semester.

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1. (02/03) Introduction and informal overview

The notes for this lecture will be much more detailed than usual, since it departs from our textbook.

1. Administrative information.

The main course page is math.mit.edu/18.675. It contains links to textbooks, the office hours schedule, and key dates. It links to this document which will summarize the main topics covered, lecture by lecture. It also links to the MIT Stellar page where the course policy, homework, and additional resources will be posted.

2. The goal of this lecture was to give an informal overview of some of the topics of the class. We say $Z$ is a (one-dimensional) standard gaussian or standard normal, denoted $Z \sim N(0, 1)$, if $Z$ is a real-valued random variable with probability density

$$g(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right).$$

Suppose $X, X_i$ are i.i.d. symmetric random signs, and $S_n = X_1 + \ldots + X_n$. Then (from 18.675) we know

$$\frac{S_n}{\sqrt{n}} \xrightarrow{d} N(0, 1).$$

(1)

This follows from the central limit theorem, or by a direct combinatorial calculation (the “DeMoivre–Laplace central limit theorem”; see [Dur19, §3.1]).

3. Now consider the simple random walk $(S_n)_{n \geq 0}$. Because of (1), over time $n$ the walk typically covers distance $\Theta(\sqrt{n})$. Suppose then that we scale space by $n$ while scaling time by $\sqrt{n}$: for any fixed $t \geq 0$,

$$X^{(n)}(t) = \frac{S_{\lfloor nt \rfloor}}{\sqrt{n}} \xrightarrow{d} N(0, t).$$

In fact the entire path of $X^{(n)}$ (a process indexed by $t$) converges in law (with respect to a suitable topology on the space of paths, which we will discuss):

$$X^{(n)} = \left( X^{(n)}(t) \right)_{t \geq 0} \xrightarrow{d} (B_t)_{t \geq 0} = B.$$

(2)

The limiting object $B$ is a (one-dimensional) standard Brownian motion.

4. Some long-term goals of this class include the rigorous construction of Brownian motion, the convergence of natural (discrete-time) processes to Brownian motion (“functional central limit theorems”), and calculations with Brownian motion. In this lecture we will see an informal sampling of these topics — namely, a sketch of Ito’s formula, and its application to the conformal invariance of the trace of planar Brownian motion (and the computation of hitting probabilities). First note that, although we have not formally defined what is a Brownian motion $B$, it is clear that for (2) to hold in any reasonable topology, $B$ will have to satisfy the following properties: (i) $B_0 = 0$; (ii) for any $0 \leq s \leq t$ we have $B_t - B_s \sim N(0, t - s)$; (iii) for any $0 \leq s_1 \leq t_1 \leq s_2 \leq t_2$ the increments $B_{t_1} - B_{s_1}$ and $B_{t_2} - B_{s_2}$ are independent from one another.
5. Itô’s formula: this is a central result in this subject, and will take us some time to prove. The informal version is as follows: suppose \( B \) is a standard Brownian motion, and \( X = (X_t)_{t \geq 0} \) evolves according to
\[
d X_t = \mu_t \, dt + \sigma_t \, dB_t
\]  
where \( \mu_t \) is the drift and \( \sigma_t \) is the local volatility. (It will be some time before we have a formal understanding of (3), but for now we can interpret it as saying that \( X_{t+dt} - X_t \) is gaussian with mean \( \mu_t \, dt \) and variance \( (\sigma_t)^2 \, dt \).

Itô’s formula characterizes the evolution of \( f(X_t) \) for \( f : \mathbb{R} \to \mathbb{R} \) twice differentiable. It starts with the second-order Taylor expansion
\[
d f(X_t) = f'(X_t) \, dX_t + \left[ \frac{f''(X_t)(dX_t)^2}{2} \right].
\]  
If \( X \) were a deterministic process with nice trajectory, then the second-order term is negligible, and we would have simply \( d f(X_t) = f'(X_t) \, dX_t \). However, the \( X \) of (3) is a stochastic process (due to the randomness of \( B \)), and it turns out that the second term above is no longer negligible: instead it turns out that
\[
\frac{(dX_t)^2}{2} = \left[ \frac{\mu_t \, dt}{2} + \mu_t \, \sigma_t \, dt \, dB_t \right] + \left[ \frac{(\sigma_t \, dB_t)^2}{2} \right] = \frac{(\sigma_t)^2 \, dt}{2}.
\]  
This is often referred to as the Itô correction.

6. For the application, we briefly review some complex analysis. Suppose \( f : D \to \mathbb{C} \) for some open domain \( D \subseteq \mathbb{C} \). We say \( f \) is holomorphic (or complex-differentiable) at \( z \in D \) if
\[
f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \in \mathbb{C}
\]  
exists, where \( h \to 0 \) in \( \mathbb{C} \). We can regard \( f \) as a mapping \( \mathbb{R}^2 \to \mathbb{R}^2 \), but being a complex-differentiable (holomorphic) mapping \( \mathbb{C} \to \mathbb{C} \) is a much stronger condition than being a real-differentiable mapping \( \mathbb{R}^2 \to \mathbb{R}^2 \). To see this, write \( u = \text{Re} \, f \), \( v = \text{Im} \, f \), so \( f = u + iv \). We also write \( f(x, y) \equiv f(x + iy) \) for \( x, y \in \mathbb{R} \).

If \( f \) is holomorphic at \( z = x + iy \), then
\[
f'(z) = \lim_{h \to 0, \text{Re} \, h \in \mathbb{R}} \frac{f(z+h) - f(z)}{h} = \frac{\partial f}{\partial x} = f_x = u_x + iv_x,
\]
\[
f'(z) = \lim_{h \to 0, \text{Im} \, h \in \mathbb{R}} \frac{f(z+h) - f(z)}{h} = \frac{\partial f}{\partial y} = f_y = u_y + iv_y.
\]  
Comparing the two calculations gives the Cauchy–Riemann equations: \( u_x = v_y \) and \( u_y = -v_x \). An immediate consequence of the Cauchy–Riemann equations is that
\[
\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u_{xx} + u_{yy} = v_{xy} - v_{yx} = 0,
\]  
i.e., \( u \) is harmonic. It follows by symmetry that \( v \) is also harmonic. Finally, for open domains \( D, D' \subseteq \mathbb{C} \), we say that \( f : D \to D' \) is a conformal mapping if \( f \) is holomorphic, and has an inverse \( f^{-1} : D' \to D \) that is also holomorphic. In particular, in this case we must have \( f'(z) \neq 0 \) for all \( z \in D \).

7. Let \((X_t, Y_t)\) be a standard Brownian motion in \( \mathbb{R}^2 \), meaning that \( X \) and \( Y \) are independent one-dimensional standard Brownian motions. Then \( Z_t = X_t + iY_t \) is a standard Brownian motion in \( \mathbb{C} \). Suppose \( f : D \to D' \) is a
conformal mapping, and write \( u = \text{Re} f, v = \text{Im} f \). By Itô’s formula (a two-dimensional generalization), as long as \( Z_t \in D \) we have

\[
du(Z_t) = \begin{pmatrix} u_x(Z_t) & u_y(Z_t) \end{pmatrix} \begin{pmatrix} dX_t \\ dY_t \end{pmatrix} + \frac{1}{2} \begin{pmatrix} u_x(Z_t) & u_y(Z_t) \\ u_y(Z_t) & u_y(Z_t) \end{pmatrix} \begin{pmatrix} dX_t \\ dY_t \end{pmatrix}
\]

\[
= \begin{pmatrix} u_x(Z_t) & u_y(Z_t) \end{pmatrix} \begin{pmatrix} dX_t \\ dY_t \end{pmatrix} + \frac{1}{2} \Delta u(Z_t) \, dt,
\]

since \( u \) is harmonic. A similar calculation applies for \( v(Z_t) \), so altogether

\[
\begin{pmatrix} du(Z_t) \\ dv(Z_t) \end{pmatrix} = \begin{pmatrix} u_x(Z_t) & u_y(Z_t) \\ v_x(Z_t) & v_y(Z_t) \end{pmatrix} \begin{pmatrix} dX_t \\ dY_t \end{pmatrix} = |f'(Z_t)| \, O(Z_t) \begin{pmatrix} dX_t \\ dY_t \end{pmatrix},
\]

where, by the Cauchy–Riemann equations, \( O(Z_t) \) is a 2 \times 2 orthogonal matrix. In fact it has determinant 1 so it is a rotation matrix. Brownian motion is invariant (in law) under orthogonal transformations: \( OB \) is equidistributed as \( B \). The scaling has the same effect as a time change: \( \sigma B_t \) is equidistributed as \( B_{\sigma^2 t} \). Thus (7) shows (at least heuristically) that \( f(Z_t) = u(Z_t) + iv(Z_t) \) is also a Brownian motion in \( C \), only with a time change determined by \( |f'(Z_t)| \). This fact is common referred to as “conformal invariance of the trace of planar Brownian motion.”

8. A question that we can answer with the above result: consider \((\varepsilon Z) \times (\varepsilon Z_{\geq 0})\) for small \( \varepsilon \). Start a simple random walk near \((0, y)\), and stop it upon hitting \((\varepsilon Z) \times \{0\}\). What is the law of the hitting location? We can approximate this by Brownian motion \( \tilde{Z}_t \) in the upper half place \( H \subseteq C \), started at \( iy \) and stopped at the first time \( \tau \) that it hits \( R \). A conformal mapping \( f : H \rightarrow D \) (the unit disc in \( C \)) that sends \( iy \mapsto 0 \) is given by

\[
f(z) = \frac{i - z/y}{i + z/y}.
\]

Let \( Y_t \) be a Brownian motion in \( D \), started from the origin and stopped at the first time \( \sigma \) that it hits the unit circle \( \partial D \). By the conformal invariance property,

\[
P\left(Z_\tau \in [a, b]\right) = P\left(Y_\sigma \in f([a, b])\right) = \frac{\int_0^b |f'(x)| \, dx}{2\pi} = \int_a^b \frac{1}{\pi x^2 + y^2} \, dx.
\]

The function \( P_y(x) \) is the Poisson kernel for the upper half plane. It coincides with the Cauchy distribution on the real line (the standard Cauchy corresponds to \( y = 1 \)). The calculation (8) is directly related to the fact that \( P_y(x) \) can be used to solve the Dirichlet boundary value problem on \( H \): if \( b : \mathbb{R} \rightarrow \mathbb{R} \) is nice enough, then its unique harmonic interpolation to \( H \) is given by

\[
h(x + iy) = \mathbb{E} \left[ b(Z_\tau) \bigg| Z_0 = x + iy \right] = \mathbb{E} \left[ b(Z_\tau + x) \bigg| Z_0 = iy \right] = \int_{\mathbb{R}} P_y(s) b(s + x) \, ds
\]

\[
= \int_{\mathbb{R}} P_y(-s) b(x - s) \, ds = \int_{\mathbb{R}} P_y(s) b(x - s) \, ds = (b \ast P_y)(x),
\]

having used that \( P_y : \mathbb{R} \rightarrow \mathbb{R} \) is an even function.

9. It is in fact possible to compute exact hitting probabilities for simple random walk \((\varepsilon Z) \times (\varepsilon Z_{\geq 0})\) (without passing to the limit \( \varepsilon \downarrow 0 \)), but this is less straightforward. One way to do it is as follows (this discussion follows [Sp87, Ch. III]). First, without loss of generality take \( \varepsilon = 1 \). Consider a simple random walk in \( Z \times Z_{\geq 0} \), started at \((0, k)\) and stopped at the first time that it hits \( \mathbb{Z} \times \{0\} \). Let \( \nu_k \) be the law of the \( x \)-coordinate at the stopping location, and let \( \zeta^{(k)} \sim \nu_k \). In order to reach \( \mathbb{Z} \times \{0\} \) from \((0, k)\), the walk must first hit \( \mathbb{Z} \times \{k - 1\} \), then \( \mathbb{Z} \times \{k - 2\} \), and so on until \( \mathbb{Z} \times \{0\} \). It follows that

\[
\zeta^{(k)} = \zeta_1 + \ldots + \zeta_k
\]

where \( \zeta, \zeta_i \) are i.i.d. samples from \( \nu_1 \). Thus, to understand the law of \( \zeta^{(k)} \) it suffices to understand the law of \( \zeta \). Denote the characteristic function \( \varphi(\theta) = \mathbb{E} [\exp(i\theta \zeta)] \). If we consider the walk started from \((0, 1)\) and
condition on the first step of the walk, we find
\[ \phi(\theta) = \mathbb{E}[\exp(i\theta \zeta)] = \frac{1}{4} \left\{ \phi(\theta)^2 + 1 + \phi(\theta)e^{i\theta} + \phi(\theta)e^{-i\theta} \right\} \]

— the four terms on the right-hand side correspond respectively to the first step being up, down, right, left.

Solving the quadratic equation gives
\[ \phi(\theta) = 2 - \cos \theta - \sqrt{(1 - \cos \theta)(3 - \cos \theta)} , \]

which is a $2\pi$-periodic function of $\theta \in \mathbb{R}$. The law $\nu_1$ can then be recovered by inverting the Fourier transform:

by [Dur19, Exercise 3.3.2], for all $x \in \mathbb{R}$ we have
\[ \nu_1(\{x\}) = \mathbb{P}(\zeta = x) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} e^{-i\theta x} \phi(\theta) \, d\theta = \int_{0}^{2\pi} e^{-i\theta x} \phi(\theta) \, d\theta . \]

For instance, $\nu_1(\{0\}) = 1 - 2/\pi \approx 0.36$. As noted above, the law $\nu_k$ is simply the $k$-fold convolution of $\nu_1$, $\nu_k = \nu_1 * \ldots * \nu_1 = (\nu_1)^{\otimes k}$. To recover the previous calculation, note that if we run simple random walk on $(n^{-1}\mathbb{Z}) \times (n^{-1}\mathbb{Z}_{\geq 0})$ started from $(0, \lfloor ny \rfloor)$ and stopped upon hitting $(n^{-1}\mathbb{Z}) \times \{0\}$, then the $x$-coordinate of the stopping location is equidistributed as
\[ \frac{\zeta_{\lfloor ny \rfloor}}{n} = \frac{1}{n} \sum_{i=1}^{\lfloor ny \rfloor} \zeta_i . \]

This has characteristic function $\phi(\theta/n)^{\lfloor ny \rfloor}$. Taking the limit $n \to \infty$, we find
\[ \lim_{n \to \infty} \left\{ \phi \left( \frac{\theta}{n} \right) \right\}^{\lfloor ny \rfloor} = e^{-y|\theta|} = \int e^{i\theta x} \frac{y}{\pi(x^2 + y^2)} \, dx = \int e^{i\theta x} p_y(x) \, dx , \]

which recovers the previous calculation (8).

Reading: beginning of [LG16, Ch. 1]. Further references: for general complex analysis see [SS03a]. For the Dirichlet boundary value problem and the Poisson kernel see [SS03b, SS03a, LL01].

2. (02/05) GAUSSIAN VARIABLES AND GAUSSIAN PROCESSES

Reading: [LG16, Ch. 1].

References


