Keep completing the square!

Suppose someone hands you a quadratic polynomial in several variables, such as

$$x^2 + 2xy - 2xz + 2y^2 + 2yz + 6z^2 - z + 1,$$

and asks you to check whether it is always ≥ 0. How do you do it?

The trick to this is a slight generalization of the high school procedure known as "completing the square", which I like to call "keep completing the square" (I stumbled on this method after meditating on what the Cholesky decomposition really *meant* in terms of quadratic polynomials). We start by trying to write down a square that agrees with our polynomial at least as far as x is concerned, that is, we try to solve the equation

$$(x + Ay + Bz + C)^2 = x^2 + 2xy - 2xz + ...,$$

for A, B, C (and ignoring the ..., since it doesn't involve x). In this case, we can take $A = 1, B = -1, C = 0$, and we get

$$(x + y - z)^2 = x + 2xy - 2xz + y^2 - 2yz + z^2.$$

Since that doesn't completely match our polynomial, we look at the difference:

$$(x^2 + 2xy - 2xz + 2y^2 + 2yz + 6z^2 - z + 1) - (x + y - z)^2 = y^2 + 4yz + 5z^2 - z + 1.$$

Now we complete the square again, this time with y, and so on. Writing the whole process in one string of equalities, we get

$$\begin{aligned} x^2 + 2xy - 2xz + 2y^2 + 2yz + 6z^2 - 2z + 1 &= (x + y - z)^2 + y^2 + 4yz + 5z^2 - z + 1 \\ &= (x + y - z)^2 + (y + 2z)^2 + z^2 - z + 1 \\ &= (x + y - z)^2 + (y + 2z)^2 + (z - \tfrac{1}{2})^2 + \tfrac{3}{4}, \end{aligned}$$

and this is clearly positive, since it is a sum of squares.

Let's do a more complicated example (the previous example was clearly chosen to let you avoid taking any square roots). What if we are faced with something like

$$6x^2 - 4xy + 2xz + 3y^2 - 4yz + 2z^2?$$

At the very first step, it seems like we'll have to take the square root of 6. What a mess! Here's how to avoid the mess: instead of starting with a square like

$$(\sqrt{6}x + Ay + Bz)^2,$$

instead we start by looking for something like

$$6(x + Ay + Bz)^2.$$

Now we can find A, B by simple division, and we get $A = -\frac{1}{3}, B = \frac{1}{6}$. Continuing, we get

$$\begin{aligned} 6x^2 - 4xy + 2xz + 3y^2 - 4yz + 2z^2 &= 6(x - \tfrac{1}{3}y + \tfrac{1}{6}z)^2 + \tfrac{7}{3}y^2 - \tfrac{10}{3}yz + \tfrac{11}{6}z^2 \\ &= 6(x - \tfrac{1}{3}y + \tfrac{1}{6}z)^2 + \tfrac{7}{3}(y - \tfrac{5}{7}z)^2 + \tfrac{9}{14}z^2, \end{aligned}$$

which is again obviously positive since it has been written as a sum of squares with positive coefficients. (By the way, I came up this polynomial by expanding out $(x-y)^2 + (x+y-z)^2 + (2x-y+z)^2$ - so we see that there can be multiple ways to write the same polynomial as a sum of squares. If we had processed the variables in a different order, we could come up with yet another way to write it as a sum of squares!)

What happens if we try to do this to a quadratic polynomial which *isn't* always ≥ 0? Obviously, something has to go wrong. Let's try the polynomial

$$x^2 - 4xy + 2xz + y^2 - 2yz + 2z^2.$$

The first step goes just fine: we get

$$x^2 - 4xy + 2xz + y^2 - 2yz + 2z^2 = (x - 2y + z)^2 - 3y^2 + 2yz + z^2.$$

But now we have a problem: the coefficient of y^2 is negative. Could our polynomial still be ≥ 0? Maybe the z^2 and the $(x-2y+z)^2$ somehow always conspire to be larger than $3y^2$? Nope! To see why, just set z to 0, and choose x to make $x-2y+z$ equal to 0, for instance, take $z = 0, y = 1, x = 2$.

In the previous example, we had a problem because the coefficient of y^2 was negative. What if the coefficient of y^2 comes out to exactly 0? For an example, let's consider the polynomial

$$x^2 - 2xy - 2xz + y^2 - 2yz + 10z^2.$$

After the first step, we get

$$x^2 - 2xy - 2xz + y^2 - 2yz + 2z^2 = (x - y - z)^2 - 4yz + 9z^2.$$

To show that this sometimes goes negative, we will take z to be whatever nonzero value we like - say, take $z = 1$ - and then pick y to make $-4yz + 9z^2$ come out negative (we can do this since, for any fixed nonzero z, $-4yz + 9z^2$ is a linear function of y with a nonzero y-coefficient), and finally pick x to make $x - y - z$ equal to 0. For instance, we can take $z = 1, y = 3, x = 4$.

At the end of the day, we have a procedure that starts with a quadratic polynomial in any number of variables, and either writes it as a sum of squares with positive coefficients, or spits out a point where it is negative! We summarize in the following theorem.

Theorem. *Suppose that $Q(x_1, ..., x_n) = \sum_{i,j} a_{ij}x_ix_j + \sum_i a_ix_i + a$, where a_{ij}, a_i, a are some coefficients. Then either we can write Q in the form*

$$Q(x_1, ..., x_n) = \sum_{i=1}^{n} c_i(x_i + b_{i(i+1)}x_{i+1} + \cdots + b_{in}x_n + b_i)^2 + c$$

with $c_i \geq 0$ for all i and $c \geq 0$, or else we can find a point $(x_1, ..., x_n)$ such that $Q(x_1, ..., x_n) < 0$.

In the case of homogeneous quadratic polynomials, people often like to represent their coefficients in a symmetric matrix. In the three variable case, the matrix

$$\begin{bmatrix} a & b & d \\ b & c & e \\ d & e & f \end{bmatrix}$$

corresponds to the polynomial

$$ax^2 + 2bxy + cy^2 + 2dxz + 2eyz + fz^2.$$

Why the random factors of 2? This is because we have the nice formula

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a & b & d \\ b & c & e \\ d & e & f \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = ax^2 + 2bxy + cy^2 + 2dxz + 2eyz + fz^2.$$

When we follow the "keep completing the square" procedure for this general three variable homogeneous quadratic, we get

$$\begin{aligned} ax^2 + 2bxy + cy^2 + 2dxz + 2eyz + fz^2 &= a(x + \tfrac{b}{a}y + \tfrac{d}{a}z)^2 + \tfrac{ac-b^2}{a}y^2 + 2\tfrac{ae-bd}{a}yz + \tfrac{af-d^2}{a}z^2 \\ &= a(x + \tfrac{b}{a}y + \tfrac{d}{a}z)^2 + \tfrac{ac-b^2}{a}(y + \tfrac{ae-bd}{ac-b^2}z)^2 + \tfrac{(af-d^2)(ac-b^2)-(ae-bd)^2}{a(ac-b^2)}z^2 \\ &= a(x + \tfrac{b}{a}y + \tfrac{d}{a}z)^2 + \tfrac{ac-b^2}{a}(y + \tfrac{ae-bd}{ac-b^2}z)^2 + \tfrac{acf+2bde-ae^2-b^2f-cd^2}{ac-b^2}z^2. \end{aligned}$$

Curiously, the coefficients in that last formula happen to be ratios of determinants:

$$\begin{aligned} \det \begin{bmatrix} a \end{bmatrix} &= a, \\ \det \begin{bmatrix} a & b \\ b & c \end{bmatrix} &= ac - b^2, \\ \det \begin{bmatrix} a & b & d \\ b & c & e \\ d & e & f \end{bmatrix} &= acf + 2bde - ae^2 - b^2f - cd^2. \end{aligned}$$

So we've proved that a three variable homogeneous quadratic is ≥ 0 if those three determinants are all positive!

Exercise. Generalize this determinant formula to any number of variables.