Minimal Taylor Algebras

Zarathustra Brady
Taylor algebras

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Definition
An idempotent algebra is *Taylor* if the variety it generates does not contain a two element set.

- All algebras in this talk will be idempotent, so I won’t mention idempotence further.
Useful facts about Taylor algebras

- Theorem (Bulatov and Jeavons)
  
  A finite algebra $\mathbb{A}$ is Taylor iff there is no set in $HS(\mathbb{A})$.  

- Theorem (Barto and Kozik)

  A finite algebra $\mathbb{A}$ is Taylor iff for every number $n$ such that every prime factor of $n$ is greater than $|\mathbb{A}|$, there is an $n$-ary cyclic term $c$, i.e. $c(x_1, x_2, \ldots, x_n) \approx c(x_2, \ldots, x_n, x_1)$.

- Corollary

  A finite algebra is Taylor iff it has a 4-ary term $t$ satisfying the identity $t(x, x, y, z) \approx t(y, z, z, x)$. 

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  A finite algebra is Taylor iff it has a 4-ary term $t$ satisfying the identity
  
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Minimal Taylor algebras

- My interest in Taylor algebras comes from the study of CSPs.

Definition

An algebra is a minimal Taylor algebra if it is Taylor, and has no proper reduct which is Taylor.

Proposition

Every finite Taylor algebra has a reduct which is a minimal Taylor algebra.

Proof.

There are only finitely many 4-ary terms $t$ which satisfy $t(x,x,y,z) \approx t(y,z,z,x)$. 

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Every finite Taylor algebra has a reduct which is a minimal Taylor algebra.

Proof.
There are only finitely many 4-ary terms $t$ which satisfy $t(x, x, y, z) \approx t(y, z, z, x)$. 


First hints of a nice theory

Theorem

If $\mathbb{A}$ is a minimal Taylor algebra, $\mathbb{B} \in HSP(\mathbb{A})$, $S \subseteq \mathbb{B}$, and $t$ a term of $\mathbb{A}$ satisfy

- $S$ is closed under $t$,
- $(S, t)$ is a Taylor algebra,

then $S$ is a subalgebra of $\mathbb{B}$, and is also a minimal Taylor algebra.
Theorem

If \( A \) is a minimal Taylor algebra, \( B \in HSP(A) \), \( S \subseteq B \), and \( t \) a term of \( A \) satisfy

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Choose $p$ a prime bigger than $|\mathbb{A}|$ and $|S|$.
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First hints of a nice theory

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If $A$ is a minimal Taylor algebra, $B \in HSP(A)$, $S \subseteq B$, and $t$ a term of $A$ satisfy

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then $S$ is a subalgebra of $B$, and is also a minimal Taylor algebra.

Choose $p$ a prime bigger than $|A|$ and $|S|$.

Choose $c$ a $p$-ary cyclic term of $A$, $u$ a $p$-ary cyclic term of $(S, t)$.

Then

$$f = c(u(x_1, x_2, \ldots, x_p), u(x_2, x_3, \ldots, x_1), \ldots, u(x_p, x_1, \ldots, x_{p-1}))$$

is a cyclic term of $A$. 
First hints of a nice theory

▶ **Theorem**  
If $\mathbb{A}$ is a minimal Taylor algebra, $\mathbb{B} \in HSP(\mathbb{A})$, $S \subseteq \mathbb{B}$, and $t$ a term of $\mathbb{A}$ satisfy

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then $S$ is a subalgebra of $\mathbb{B}$, and is also a minimal Taylor algebra.

▶ Choose $p$ a prime bigger than $|\mathbb{A}|$ and $|S|$.
▶ Choose $c$ a $p$-ary cyclic term of $\mathbb{A}$, $u$ a $p$-ary cyclic term of $(S, t)$.
▶ Then

$$f = c(u(x_1, x_2, \ldots, x_p), u(x_2, x_3, \ldots, x_1), \ldots, u(x_p, x_1, \ldots, x_{p-1}))$$

is a cyclic term of $\mathbb{A}$.
▶ Have $f|_S = u|_S$ by idempotence.
A few consequences

- **Proposition**
  
  For \( \mathbb{A} \) minimal Taylor, \( a, b \in \mathbb{A} \), then \( \{a, b\} \) is a semilattice subalgebra of \( \mathbb{A} \) with absorbing element \( b \) iff

\[
\begin{bmatrix}
  b \\
  b
\end{bmatrix} \in Sg_{\mathbb{A}^2} \left\{ \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} b \\ a \end{bmatrix} \right\}.
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  \]

- **Proposition**
  
  For \( A \) minimal Taylor, \( a, b \in A \), then \( \{a, b\} \) is a majority subalgebra of \( A \) iff
  
  \[
  \begin{bmatrix}
  a & b \\
  a & b \\
  a & b
  \end{bmatrix} \in S_{g_A}^{3 \times 2} \left\{ \begin{bmatrix} a & b \\ a & b \\ b & a \end{bmatrix}, \begin{bmatrix} a & b \\ b & a \\ a & b \end{bmatrix}, \begin{bmatrix} b & a \\ a & b \\ a & b \end{bmatrix} \right\}.
  \]
A few consequences, ctd.

- Proposition

For $\mathbb{A}$ minimal Taylor, $a, b \in \mathbb{A}$, then $\{a, b\}$ is a $\mathbb{Z}/2^\text{aff}$ subalgebra of $\mathbb{A}$ iff

$$
\begin{bmatrix}
    b & a \\
    b & a \\
    b & a
\end{bmatrix}
\in Sg_{\mathbb{A}^{3 \times 2}} \left\{ \begin{bmatrix}
    a & b \\
    a & b \\
    b & a
\end{bmatrix}, \begin{bmatrix}
    a & b \\
    b & a \\
    a & b
\end{bmatrix}, \begin{bmatrix}
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  b & a \\
  b & a \\
  \end{bmatrix}
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  a & b \\
  a & b \\
  b & a \\
  \end{bmatrix}, \begin{bmatrix}
  a & b \\
  b & a \\
  a & b \\
  \end{bmatrix}, \begin{bmatrix}
  b & a \\
  a & b \\
  a & b \\
  \end{bmatrix} \right\}.
  \]

- If there is an automorphism of \( \mathbb{A} \) which interchanges \( a, b \), then we only have to consider

  \[
  Sg_{\mathbb{A}^{3}} \left\{ \begin{bmatrix}
  a \\
  a \\
  b \\
  \end{bmatrix}, \begin{bmatrix}
  a \\
  b \\
  a \\
  \end{bmatrix}, \begin{bmatrix}
  b \\
  a \\
  a \\
  \end{bmatrix} \right\}.
  \]
Daisy Chain Terms

- It’s difficult to write down explicit examples without nice terms.
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- Choose a $p$-ary cyclic term $c$. 

\[
w(x, y, z) = c(x, \ldots, x, a, y, \ldots, y, p-2a, z, \ldots, z)
\]

- This satisfies $w(x, x, y) \approx w(y, x, x)$. 
- Also have $w(x, y, x) = c(x, \ldots, x, a, y, \ldots, y, p-2a, x, \ldots, x)$. 
Daisy Chain Terms

- It’s difficult to write down explicit examples without nice terms.

- Choose a \( p \)-ary cyclic term \( c \).

- For any \( a < \frac{p}{2} \), can make a ternary term \( w(x, y, z) \) via:

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w(x, y, z) = c_{\{x, \ldots, x, y, \ldots, y, z, \ldots, z\}_{\{a, p-2a, a\}}}.\]
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$$w(x, y, z) = c\left(x, \ldots, x, y, \ldots, y, z, \ldots, z\right).$$

$$\begin{array}{c}
\underbrace{x, \ldots, x} \\
a
\end{array} \quad \begin{array}{c}
\underbrace{y, \ldots, y} \\
p-2a
\end{array} \quad \begin{array}{c}
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\end{array}$$

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  \[a \quad p-2a \quad a\]

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- Also have

  $$w(x, y, x) = c(x, \ldots, x, y, \ldots, y, x, \ldots, x).$$

  \[a \quad p-2a \quad a\]
Daisy Chain Terms, ctd.

- From a sequence 

\[ a, p - 2a, p - 2(p - 2a), \ldots \]

we get a sequence of ternary terms:

\[ w_0(x, x, y) \approx w_0(y, x, x) \approx w_1(x, y, x), \]
\[ w_1(x, x, y) \approx w_1(y, x, x) \approx w_2(x, y, x), \]
\[ \vdots \]

If \( p \) is large enough and \( a \) is close enough to \( p^3 \), then the sequence can become arbitrarily long. Since there are only finitely many ternary functions in \( \text{Clo}(A) \), we eventually get a cycle.
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What do they mean?

- How can daisy chain terms be useful to us?
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- For \( a, b \in A \), define a binary relation \( D_{ab} \subseteq A^2 \) by

\[
D_{ab} = \left\{ \begin{bmatrix} c \\ d \end{bmatrix} \text{ s.t. } \begin{bmatrix} c \\ d \\ c \end{bmatrix} \in Sg_{A^3} \left\{ \begin{bmatrix} a \\ a \\ b \end{bmatrix}, \begin{bmatrix} a \\ b \\ a \end{bmatrix}, \begin{bmatrix} b \\ a \\ a \end{bmatrix} \right\} \right\}.
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- If $\begin{bmatrix} a \\ a \end{bmatrix} \in D_{ab}$ and there is an automorphism interchanging $a, b$, then $\{a, b\}$ is a majority algebra.
What do they mean?

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- Proposition

  *If $\mathbb{A}$ has daisy chain terms and $a, b \in \mathbb{A}$, then if we consider $D_{ab}$ as a digraph, it must contain a directed cycle.*
Describing a minimal Taylor algebra

- If $p = w_i$, $q = w_{i+1}$ are any pair of adjacent daisy chain terms, then they satisfy the system

\[ p(x, x, y) \approx p(y, x, x) \approx q(x, y, x), \]
\[ q(x, x, y) \approx q(y, x, x). \]
Describing a minimal Taylor algebra

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  p(x, x, y) \approx p(y, x, x) \approx q(x, y, x), \\
  q(x, x, y) \approx q(y, x, x).
  \]

- Thus $p, q$ generate a Taylor clone, so $\text{Clo}(A) = \langle p, q \rangle$ if $A$ is minimal Taylor.
Describing a minimal Taylor algebra

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- In particular, the number of minimal Taylor clones on a set of \( n \) elements is at most \( n^{2n^3} \).
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- **Conjecture**

  Every minimal Taylor clone can be generated by a *single* ternary function.
Daisy chain terms in the basic algebras

- Proposition
  
  If \( w_i \) are daisy chain terms and \( A \) is a semilattice, then each \( w_i \) is the symmetric ternary semilattice operation on \( A \).
Daisy chain terms in the basic algebras

Proposition

If \( w_i \) are daisy chain terms and \( \mathbb{A} \) is a semilattice, then each \( w_i \) is the symmetric ternary semilattice operation on \( \mathbb{A} \).

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If \( w_i \) are daisy chain terms and \( \mathbb{A} \) is a majority algebra, then each \( w_i \) is a majority operation on \( \mathbb{A} \).
Daisy chain terms in the basic algebras

**Proposition**

If \( w_i \) are daisy chain terms and \( \mathbb{A} \) is a semilattice, then each \( w_i \) is the symmetric ternary semilattice operation on \( \mathbb{A} \).

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If \( w_i \) are daisy chain terms and \( \mathbb{A} \) is a majority algebra, then each \( w_i \) is a majority operation on \( \mathbb{A} \).

**Proposition**

If \( w_i \) are daisy chain terms and \( \mathbb{A} \) is affine, then there is a sequence \( a_i \) such that \( w_i \) is given by

\[
w_i(x, y, z) = a_ix + (1 - 2a_i)y + a_iz,
\]

with \( a_{i+1} = 1 - 2a_i \).

If \( a_0 = 0 \), then \( w_1 \) is the Mal’cev operation \( x - y + z \) and \( w_{-1} \) is the operation \( \frac{x + z}{2} \).
Bulatov’s graph

- Bulatov studies finite Taylor algebras via three types of edges: semilattice, majority, and affine.
Bulatov’s graph

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- In minimal Taylor algebras, we can define his edges more simply.

Definition

If $A$ is minimal Taylor and $a, b \in A$, then $(a, b)$ is an edge if there is a congruence $\theta$ on $Sg\{a, b\}$ s.t. $Sg\{a, b\}/\theta$ is isomorphic to either a two-element semilattice, a two element majority algebra, or an affine algebra.
Bulatov’s graph

Bulatov studies finite Taylor algebras via three types of edges: semilattice, majority, and affine.

In minimal Taylor algebras, we can define his edges more simply.

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If $\mathbb{A}$ is minimal Taylor and $a, b \in \mathbb{A}$, then $(a, b)$ is an edge if there is a congruence $\theta$ on $Sg\{a, b\}$ s.t.

$$Sg\{a, b\}/\theta$$

is isomorphic to either a two-element semilattice, a two element majority algebra, or an affine algebra.
Connectivity

- **Theorem (Bulatov)**

  *If $A$ is minimal Taylor, then the associated graph is connected.*
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- We can simplify the proof!
- If $\mathbb{A}$ is a minimal counterexample:
  - the hypergraph of proper subalgebras must be disconnected,
  - $\mathbb{A}$ is generated by two elements $a, b$, and
  - $\mathbb{A}$ has no proper congruences.
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- It’s not hard to show there must be an automorphism interchanging $a, b.$
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  - \( A \) has no proper congruences.
- It’s not hard to show there must be an automorphism interchanging \( a, b \).
- Consider the binary relation \( D_{ab} \)! 
Connectivity, ctd.

- Recall the definition of $\mathcal{D}_{ab}$:

$$
\mathcal{D}_{ab} = \left\{ \begin{bmatrix} c \\ d \\ c \end{bmatrix} \text{ s.t. } \begin{bmatrix} c \\ d \\ c \end{bmatrix} \in Sg_{A^3} \left\{ \begin{bmatrix} a \\ b \\ a \end{bmatrix}, \begin{bmatrix} a \\ b \\ a \end{bmatrix}, \begin{bmatrix} b \\ a \\ a \end{bmatrix} \right\} \right\}.
$$
Connectivity, ctd.

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$$

- Have $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{D}_{ab}$, want to show that either $\begin{bmatrix} a \\ a \end{bmatrix} \in \mathbb{D}_{ab}$ or $\mathbb{A}$ is affine.
Connectivity, ctd.

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- Have $\begin{bmatrix} a \\ b \end{bmatrix} \in D_{ab}$, want to show that either $\begin{bmatrix} a \\ a \end{bmatrix} \in D_{ab}$ or $A$ is affine.

- The daisy chain terms give us $c, d, e \in A$ such that

$$\begin{bmatrix} c \\ d \end{bmatrix}, \begin{bmatrix} d \\ e \end{bmatrix} \in D_{ab}.$$
Connectivity, ctd.

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\mathbb{D}_{ab} = \left\{ \begin{bmatrix} c \\ d \\ c \\ d \end{bmatrix} \text{ s.t. } \begin{bmatrix} c \\ d \\ c \\ d \end{bmatrix} \in Sg_{\mathbb{A}^{3}} \left\{ \begin{bmatrix} a \\ a \\ b \\ a \end{bmatrix}, \begin{bmatrix} a \\ b \\ a \\ a \end{bmatrix}, \begin{bmatrix} b \\ a \\ a \\ a \end{bmatrix} \right\} \right\}.
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$$
\begin{bmatrix} c \\ d \\ c \\ d \end{bmatrix}, \begin{bmatrix} d \\ e \\ d \\ e \end{bmatrix} \in \mathbb{D}_{ab}.
$$

- If both $Sg\{a, d\}$ and $Sg\{d, b\}$ are proper subalgebras, then the hypergraph of proper subalgebras is connected.
Connectivity, ctd.

- Recall the definition of $D_{ab}$:

$$D_{ab} = \left\{ \begin{bmatrix} c \\ d \end{bmatrix} \text{ s.t. } \begin{bmatrix} c \\ d \\ c \end{bmatrix} \in Sg_{\mathbb{A}^3} \left\{ \begin{bmatrix} a \\ b \\ a \end{bmatrix}, \begin{bmatrix} a \\ b \\ a \end{bmatrix}, \begin{bmatrix} b \\ a \\ a \end{bmatrix} \right\} \right\}. $$

- Have $\begin{bmatrix} a \\ b \end{bmatrix} \in D_{ab}$, want to show that either $\begin{bmatrix} a \\ a \end{bmatrix} \in D_{ab}$ or $\mathbb{A}$ is affine.

- The daisy chain terms give us $c, d, e \in \mathbb{A}$ such that

$$\begin{bmatrix} c \\ d \end{bmatrix}, \begin{bmatrix} d \\ e \end{bmatrix} \in D_{ab}. $$

- If both $Sg\{a, d\}$ and $Sg\{d, b\}$ are proper subalgebras, then the hypergraph of proper subalgebras is connected.

- Then we can show $D_{ab}$ is subdirect, and the proof flows naturally from here.
Can we do better?

- Can we get rid of congruences in the definition of the edges?
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- Proposition (Bulatov)
  
  For every semilattice edge from $a$ to $b$, there is a $b'$ in the congruence class of $b$ such that $\{a, b'\}$ is a two element semilattice algebra.

Similar statements fail for majority edges and affine edges.

There are minimal Taylor algebras $A, B$ of size 4 which have congruences $\theta$ such that:

- $A/\theta$ is a two element majority algebra and $B/\theta$ is $\mathbb{Z}/2$ aff,
- each congruence class of $\theta$ is a copy of $\mathbb{Z}/2$ aff,
- every proper subalgebra of $A$ or $B$ is contained in a congruence class of $\theta$,
- $A$ has a 3-edge term and $B$ is Mal’cev,
- $\theta$ is the center of $A$ or $B$ in the sense of commutator theory.
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» Can we get rid of congruences in the definition of the edges?

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Evil algebra #1

- $\mathbb{A} = (\{a, b, c, d\}, g)$, where $g$ is an idempotent ternary symmetric operation.
Evil algebra #1

\( \mathbb{A} = (\{a, b, c, d\}, g) \), where \( g \) is an idempotent ternary symmetric operation.

\( g \) commutes with the cyclic permutation \( \sigma = (a \ b \ c \ d) \) and satisfies

\[
\begin{align*}
g(a, a, b) &= a, \\
g(a, a, c) &= c, \\
g(a, a, d) &= c, \\
g(a, b, c) &= c.
\end{align*}
\]
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- \( \theta \) corresponds to the partition \( \{a, c\} , \{b, d\} \).

- The algebra \( S = Sg_{\mathbb{A}^2} \{(a, b), (b, a)\} \) has a congruence \( \psi \) corresponding to the partition
  \[
  \left\{ \begin{bmatrix} a \\ b \end{bmatrix} , \begin{bmatrix} b \\ c \end{bmatrix} , \begin{bmatrix} c \\ d \end{bmatrix} , \begin{bmatrix} d \\ a \end{bmatrix} \right\} , \left\{ \begin{bmatrix} a \\ d \end{bmatrix} , \begin{bmatrix} b \\ a \end{bmatrix} , \begin{bmatrix} c \\ b \end{bmatrix} , \begin{bmatrix} d \\ c \end{bmatrix} \right\},
  \]
  such that \( S/\psi \) is isomorphic to \( \mathbb{Z}/2^{\text{aff}} \).
Evil algebra #2

\[ \mathbb{B} = \{a, b, c, d\}, p \), where \( p \) is a Mal’cev operation.
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- $\mathcal{B} = (\{a, b, c, d\}, p)$, where $p$ is a Mal’cev operation.
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- The polynomials $+_a = p(\cdot, a, \cdot), +_b = p(\cdot, b, \cdot)$ define abelian groups:

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- $\theta$ corresponds to the partition $\{a, c\}, \{b, d\}$. 
Evil algebra #2

- \( B = (\{a, b, c, d\}, p) \), where \( p \) is a Mal’cev operation.
- \( p \) commutes with the permutations \( \sigma = (a \ c)(b \ d) \) and \( \tau = (a \ c) \).
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Zhuk’s four cases

- **Theorem (Zhuk)**
  
  If $\mathbb{A}$ is minimal Taylor, then at least one of the following holds:
  
  - $\mathbb{A}$ has a proper binary absorbing subalgebra,
  - $\mathbb{A}$ has a proper “center”,
  - $\mathbb{A}$ has a nontrivial affine quotient, or
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- **Definition**
  
  $C \leq A$ is a *center* of $A$ if there exist
  - a binary-absorption-free Taylor algebra $B$ and
  - a subdirect relation $R \leq_{sd} A \times B$, such that
  - $C = \left\{ c \in A \text{ s.t. } \forall b \in B, \begin{bmatrix} c \\ b \end{bmatrix} \in R \right\}$.
Zhuk’s four cases

▶ Theorem (Zhuk)

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▶ Theorem (Zhuk)

If $C$ is a center of $A$, then $C$ is a ternary absorbing subalgebra of $A$. 
Centers and Daisy Chain terms

Theorem

If $A$ is minimal Taylor and $M \in HSP(A)$ is the two element majority algebra on the domain $\{0, 1\}$, then the following are equivalent:

- $C$ is a ternary absorbing subalgebra of $A$,
- there is a $p$-ary cyclic term $c$ of $A$ such that whenever $\#\{x_i \in C\} > \frac{p}{2}$, we have $c(x_1, \ldots, x_p) \in C$,
- the binary relation $R \subseteq A \times M$ given by
  
  $$R = (A \times \{0\}) \cup (C \times \{0, 1\})$$

  is a subalgebra of $A \times M$,
- every daisy chain term $w_i(x, y, z)$ witnesses the fact that $C$ ternary absorbs $A$. 
Centers produce majority quotients

- If $C, D$ are centers, then for any daisy chain terms $w_i$, we must have

$$w_i(C, C, D), w_i(C, D, C), w_i(D, C, C) \subseteq C$$

and

$$w_i(C, D, D), w_i(D, C, D), w_i(D, D, C) \subseteq D,$$

so $C \cup D$ is a subalgebra of $A$. 

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\end{align*}
\]

so \( C \cup D \) is a subalgebra of \( A \).

- If \( C \cap D = \emptyset \), then the equivalence relation \( \theta \) on \( C \cup D \) with parts \( C, D \) is preserved by each daisy chain term \( w_i \), and \( (C \cup D)/\theta \) is a two element majority algebra.
Binary absorption is strong absorption

**Theorem**

*If $\mathbb{A}$ is minimal Taylor, then the following are equivalent:*

- $\mathbb{B}$ binary absorbs $\mathbb{A}$,
- there exists a cyclic term $c$ such that if any $x_i \in \mathbb{B}$, then $c(x_1, ..., x_p) \in \mathbb{B}$,
- the ternary relation $R = \{(x, y, z) \text{ s.t. } (x \not\in \mathbb{B}) \implies (y = z)\}$ is a subalgebra of $\mathbb{A}^3$,
- every term $f$ of $\mathbb{A}$ which depends on all its inputs is such that if any $x_i \in \mathbb{B}$, then $f(x_1, ..., x_n) \in \mathbb{B}$. 
Minimal Taylor algebras generated by two elements

Theorem

If \( A \) is minimal Taylor and \( A = Sg\{a, b\} \), then the following are equivalent:

\[ \begin{align*}
\text{B binary absorbs } A, \\
\text{A} = \text{B} \cup \{a, b\} \text{ and there is a congruence } \theta \text{ such that } \text{B is a congruence class of } \theta, \text{ and } A/\theta \text{ is a semilattice.}
\end{align*} \]
Minimal Taylor algebras generated by two elements

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If $\mathbb{A}$ is minimal Taylor and $\mathbb{A} = \text{Sg}\{a, b\}$, then the following are equivalent:

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2. $\mathbb{A} = \mathbb{B} \cup \{a, b\}$ and there is a congruence $\theta$ such that $\mathbb{B}$ is a congruence class of $\theta$, and $\mathbb{A}/\theta$ is a semilattice.

Theorem
If $\mathbb{A}$ is minimal Taylor and $\mathbb{A} = \text{Sg}\{a, b\}$, then $\mathbb{A}$ is not polynomially complete.
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▶ Theorem
If $\mathbb{A}$ is minimal Taylor and $\mathbb{A} = \text{Sg}\{a,b\}$, then $\mathbb{A}$ is not polynomially complete.

▶ Minimal Taylor algebras generated by two elements are nicer than general minimal Taylor algebras.
Minimal Taylor algebras generated by two elements

▶ Theorem
If $A$ is minimal Taylor and $A = Sg\{a, b\}$, then the following are equivalent:
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▶ Theorem
If $A$ is minimal Taylor and $A = Sg\{a, b\}$, then $A$ is not polynomially complete.

▶ Minimal Taylor algebras generated by two elements are nicer than general minimal Taylor algebras.
▶ It’s good enough to understand such algebras.
Big conjecture

- Conjecture
  Suppose $\mathbb{A}$ is minimal Taylor, generated by two elements $a, b$, and has no affine or semilattice quotient. Then each of $a, b$ is contained in a proper ternary absorbing subalgebra of $\mathbb{A}$.
Big conjecture

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- **Proposition**
  
  Suppose the conjecture holds. Then any daisy chain term $w_i$ which is nontrivial on every affine algebra in $HS(\mathbb{A})$ generates $\text{Clo}(\mathbb{A})$. In particular, $\text{Clo}(\mathbb{A})$ is generated by a single ternary term.
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- **Theorem (Kearnes)**
  Suppose a minimal Taylor algebra has no semilattice edges and has its clone generated by a single ternary term. Then it has a 3-edge term.
Thank you for your attention.