Coarse Classification of Binary Minimal Clones

Zarathustra Brady
Minimal clones

- A clone $C$ is \textit{minimal} if $f \in C$ nontrivial implies $C = \text{Clo}(f)$. 
Minimal clones

- A clone $C$ is *minimal* if $f \in C$ nontrivial implies $C = \text{Clo}(f)$.

- If $\text{Clo}(f)$ is minimal and $g \in \text{Clo}(f)$ nontrivial, then $f \in \text{Clo}(g)$. 

- A set $A$ is called *trivial* if all of its operations are projections. Otherwise, we say $A$ is *nontrivial*.
Minimal clones

- A clone $C$ is *minimal* if $f \in C$ nontrivial implies $C = \text{Clo}(f)$.

- If $\text{Clo}(f)$ is minimal and $g \in \text{Clo}(f)$ nontrivial, then $f \in \text{Clo}(g)$.

- $\mathbb{A}$ is called a *set* if all of its operations are projections. Otherwise, we say $\mathbb{A}$ is *nontrivial*. 
Minimal clones

- A clone \( \mathcal{C} \) is *minimal* if \( f \in \mathcal{C} \) nontrivial implies \( \mathcal{C} = \text{Clo}(f) \).

- If \( \text{Clo}(f) \) is minimal and \( g \in \text{Clo}(f) \) nontrivial, then \( f \in \text{Clo}(g) \).

- \( \mathbb{A} \) is called a *set* if all of its operations are projections. Otherwise, we say \( \mathbb{A} \) is *nontrivial*.

- If \( \text{Clo}(\mathbb{A}) \) is minimal and \( \mathbb{B} \in \text{Var}(\mathbb{A}) \) nontrivial, then \( \text{Clo}(\mathbb{B}) \) is minimal.
Rosenberg’s Five Types Theorem

Theorem (Rosenberg)
Suppose that $\mathbb{A} = (A, f)$ is a finite clone-minimal algebra, and $f$ has minimal arity among nontrivial elements of $\text{Clo}(\mathbb{A})$. Then one of the following is true:

1. $f$ is a unary operation which is either a permutation of prime order or satisfies $f(f(x)) \approx f(x)$,
2. $f$ is ternary, and $\mathbb{A}$ is the idempotent reduct of a vector space over $\mathbb{F}_2$,
3. $f$ is a ternary majority operation,
4. $f$ is a semiprojection of arity at least 3,
5. $f$ is an idempotent binary operation.
Nice properties

- We say a property $\mathcal{P}$ of functions $f$ is *nice* if it satisfies the following two conditions:
  
  1. Given $f$ as input, we can verify in polynomial time whether $f$ has property $\mathcal{P}$,
  2. If $f$ has property $\mathcal{P}$ and $g \in \text{Clo}(f)$ is nontrivial, then there is a nontrivial $f' \in \text{Clo}(g)$ such that $f'$ has property $\mathcal{P}$.

The first four cases in Rosenberg's classification are nice properties.
Nice properties

We say a property $\mathcal{P}$ of functions $f$ is *nice* if it satisfies the following two conditions:

- Given $f$ as input, we can verify in polynomial time whether $f$ has property $\mathcal{P}$,
- If $f$ has property $\mathcal{P}$ and $g \in \text{Clo}(f)$ is nontrivial, then there is a nontrivial $f' \in \text{Clo}(g)$ such that $f'$ has property $\mathcal{P}$.

The first four cases in Rosenberg's classification are nice properties.
Nice properties

We say a property $\mathcal{P}$ of functions $f$ is \textit{nice} if it satisfies the following two conditions:

- Given $f$ as input, we can verify in polynomial time whether $f$ has property $\mathcal{P}$,

- If $f$ has property $\mathcal{P}$ and $g \in \text{Clo}(f)$ is nontrivial, then there is a nontrivial $f' \in \text{Clo}(g)$ such that $f'$ has property $\mathcal{P}$.

The first four cases in Rosenberg's classification are nice properties.
We say a property $\mathcal{P}$ of functions $f$ is *nice* if it satisfies the following two conditions:

- Given $f$ as input, we can verify in polynomial time whether $f$ has property $\mathcal{P}$,
- If $f$ has property $\mathcal{P}$ and $g \in \text{Clo}(f)$ is nontrivial, then there is a nontrivial $f' \in \text{Clo}(g)$ such that $f'$ has property $\mathcal{P}$.

The first four cases in Rosenberg’s classification are nice properties.
Majority is a nice property

- As an example, we’ll check that being a ternary majority operation is a nice property.
Majority is a nice property

- As an example, we’ll check that being a ternary majority operation is a nice property.

- **Lemma**

  If $f$ is a majority operation and $g \in \text{Clo}(f)$ is nontrivial, then $g$ is a near-unanimity operation.
Majority is a nice property

As an example, we’ll check that being a ternary majority operation is a nice property.

Lemma

If \( f \) is a majority operation and \( g \in \text{Clo}(f) \) is nontrivial, then \( g \) is a near-unanimity operation.

The proof is by induction on the construction of \( g \) in terms of \( f \).
Majority is a nice property

▶ As an example, we’ll check that being a ternary majority operation is a nice property.

▶ **Lemma**

_If f is a majority operation and g ∈ Clo(f) is nontrivial, then g is a near-unanimity operation._

▶ The proof is by induction on the construction of g in terms of f.

▶ → g has a majority term as an identification minor.
Our goal is to find a list of nice properties $\mathcal{P}_1, \mathcal{P}_2, ...$ such that every minimal clone has an operation satisfying one of these nice properties.
Coarse Classification

- Our goal is to find a list of nice properties $\mathcal{P}_1, \mathcal{P}_2, \ldots$ such that every minimal clone has an operation satisfying one of these nice properties.

- We’ll call such a list a coarse classification of minimal clones.
Coarse Classification

- Our goal is to find a list of nice properties $\mathcal{P}_1, \mathcal{P}_2, \ldots$ such that every minimal clone has an operation satisfying one of these nice properties.

- We’ll call such a list a coarse classification of minimal clones.

- By Rosenberg’s result, we just need to find a coarse classification of binary minimal clones.
Coarse Classification

- Our goal is to find a list of nice properties $\mathcal{P}_1, \mathcal{P}_2, ...$ such that every minimal clone has an operation satisfying one of these nice properties.

- We’ll call such a list a *coarse classification* of minimal clones.

- By Rosenberg’s result, we just need to find a coarse classification of *binary* minimal clones.

- The main challenge is to find properties of binary operations $f$ that ensure that $\text{Clo}(f)$ doesn’t contain any semiprojections.
Theorem (Z.)

Suppose $A$ is a finite algebra which is both clone-minimal and Taylor. Then one of the following is true:

1. $A$ is the idempotent reduct of a vector space over $\mathbb{F}_p$ for some prime $p$,
2. $A$ is a majority algebra,
3. $A$ is a spiral.

The proof uses the characterization of bounded width algebras. All three cases are given by nice properties.
Theorem (Z.)

Suppose $\mathbb{A}$ is a finite algebra which is both clone-minimal and Taylor. Then one of the following is true:

1. $\mathbb{A}$ is the idempotent reduct of a vector space over $\mathbb{F}_p$ for some prime $p$,
2. $\mathbb{A}$ is a majority algebra,
3. $\mathbb{A}$ is a spiral.

The proof uses the characterization of bounded width algebras.
Theorem (Z.)

Suppose \( \mathbb{A} \) is a finite algebra which is both clone-minimal and Taylor. Then one of the following is true:

1. \( \mathbb{A} \) is the idempotent reduct of a vector space over \( \mathbb{F}_p \) for some prime \( p \),

2. \( \mathbb{A} \) is a majority algebra,

3. \( \mathbb{A} \) is a spiral.

The proof uses the characterization of bounded width algebras.

All three cases are given by nice properties.
Definition
\( \mathbb{A} = (A, f) \) is a spiral if \( f \) is binary, idempotent, commutative, and for any \( a, b \in \mathbb{A} \) either \( \{a, b\} \) is a subalgebra of \( \mathbb{A} \), or \( \text{Sg}_{\mathbb{A}} \{a, b\} \) has a surjective map to the free semilattice on two generators.

Any 2-semilattice is a (clone-minimal) spiral.

A clone-minimal spiral which is not a 2-semilattice:
Spirals

Definition
\( \mathbb{A} = (A, f) \) is a spiral if \( f \) is binary, idempotent, commutative, and for any \( a, b \in \mathbb{A} \) either \( \{a, b\} \) is a subalgebra of \( \mathbb{A} \), or \( \text{Sg}_{\mathbb{A}} \{a, b\} \) has a surjective map to the free semilattice on two generators.

- Any 2-semilattice is a (clone-minimal) spiral.
Spirals

Definition
\( \mathbb{A} = (A, f) \) is a spiral if \( f \) is binary, idempotent, commutative, and for any \( a, b \in \mathbb{A} \) either \( \{a, b\} \) is a subalgebra of \( \mathbb{A} \), or \( \text{Sg}_{\mathbb{A}} \{a, b\} \) has a surjective map to the free semilattice on two generators.

Any 2-semilattice is a (clone-minimal) spiral.

A clone-minimal spiral which is not a 2-semilattice:
The non-Taylor case

Theorem (Z.)
Suppose that $\mathbb{A} = (A, f)$ is a binary minimal clone which is not Taylor. Then, after possibly replacing $f(x, y)$ by $f(y, x)$, one of the following is true:

1. $\mathbb{A}$ is a rectangular band,
The non-Taylor case

Theorem (Z.)

Suppose that $\mathbb{A} = (A, f)$ is a binary minimal clone which is not Taylor. Then, after possibly replacing $f(x, y)$ by $f(y, x)$, one of the following is true:

1. $\mathbb{A}$ is a rectangular band,
2. there is a nontrivial $s \in \text{Clo}(f)$ which is a “partial semilattice operation”: $s(x, s(x, y)) \approx s(s(x, y), x) \approx s(x, y)$,
The non-Taylor case

Theorem (Z.)

Suppose that $\mathbb{A} = (A, f)$ is a binary minimal clone which is not Taylor. Then, after possibly replacing $f(x, y)$ by $f(y, x)$, one of the following is true:

1. $\mathbb{A}$ is a rectangular band,
2. there is a nontrivial $s \in \text{Clo}(f)$ which is a “partial semilattice operation”: $s(x, s(x, y)) \approx s(s(x, y), x) \approx s(x, y),$
3. $\mathbb{A}$ is a $p$-cyclic groupoid for some prime $p,$
The non-Taylor case

Theorem (Z.)

Suppose that $\mathbb{A} = (A, f)$ is a binary minimal clone which is not Taylor. Then, after possibly replacing $f(x, y)$ by $f(y, x)$, one of the following is true:

1. $\mathbb{A}$ is a rectangular band,
2. there is a nontrivial $s \in \text{Clo}(f)$ which is a “partial semilattice operation”: $s(x, s(x, y)) \approx s(s(x, y), x) \approx s(x, y),$
3. $\mathbb{A}$ is a $p$-cyclic groupoid for some prime $p$,
4. $\mathbb{A}$ is an idempotent groupoid satisfying $(xy)(zx) \approx xy$ (“neighborhood algebra”),
The non-Taylor case

Theorem (Z.)

Suppose that $\mathbb{A} = (A, f)$ is a binary minimal clone which is not Taylor. Then, after possibly replacing $f(x, y)$ by $f(y, x)$, one of the following is true:

1. $\mathbb{A}$ is a rectangular band,
2. there is a nontrivial $s \in \text{Clo}(f)$ which is a “partial semilattice operation”: $s(x, s(x, y)) \approx s(s(x, y), x) \approx s(x, y),$
3. $\mathbb{A}$ is a $p$-cyclic groupoid for some prime $p$,
4. $\mathbb{A}$ is an idempotent groupoid satisfying $(xy)(zx) \approx xy$ (“neighborhood algebra”),
5. $\mathbb{A}$ is a “dispersive algebra”.
Dispersive algebras: definition

We define the variety $\mathcal{D}$ of idempotent groupoids satisfying

\[ x(yx) \approx (xy)x \approx (xy)y \approx (xy)(yx) \approx xy, \quad (\mathcal{D}1) \]

\[ \forall n \geq 0 \quad x(...((xy_1)y_2) \cdots y_n)) \approx x. \quad (\mathcal{D}2) \]
Dispersive algebras: definition

- We define the variety \( \mathcal{D} \) of idempotent groupoids satisfying

\[
x(yx) \approx (xy)x \approx (xy)y \approx (xy)(yx) \approx xy,
\]

\( (\mathcal{D}1) \)

\[
\forall n \geq 0 \quad x(...((xy_1)y_2)\cdots y_n)) \approx x.
\]

\( (\mathcal{D}2) \)

- Proposition (Lévai, Pálfy)

If \( A \in \mathcal{D} \), then \( \text{Clo}(A) \) is a minimal clone. Also, \( \mathcal{F}_\mathcal{D}(x, y) \) has exactly four elements: \( x, y, xy, yx \).
Dispersive algebras: definition

- We define the variety $\mathcal{D}$ of idempotent groupoids satisfying

$$x(yx) \approx (xy)x \approx (xy)y \approx (xy)(yx) \approx xy, \quad (\mathcal{D}1)$$

$$\forall n \geq 0 \quad x(...((xy_1)y_2)\cdots y_n)) \approx x. \quad (\mathcal{D}2)$$

- Proposition (Lévai, Pálfy)

  If $\mathbb{A} \in \mathcal{D}$, then $\text{Clo}(\mathbb{A})$ is a minimal clone. Also, $\mathcal{F}_\mathcal{D}(x, y)$ has exactly four elements: $x, y, xy, yx$.

- Definition

  An idempotent groupoid $\mathbb{A}$ is dispersive if it satisfies $(\mathcal{D}2)$ and if for all $a, b \in \mathbb{A}$, either $\{a, b\}$ is a two element subalgebra of $\mathbb{A}$ or there is a surjective map

$$\text{Sg}_{\mathbb{A}^2}\{(a, b), (b, a)\} \rightarrow \mathcal{F}_\mathcal{D}(x, y).$$
Absorption identities

- An absorption identity is an identity of the form

\[ t(x_1, \ldots, x_n) \approx x_i. \]
Absorption identities

- An absorption identity is an identity of the form

  \[ t(x_1, ..., x_n) \approx x_i. \]

- If \( \mathbb{A} \) is clone-minimal and \( \mathbb{B} \in \text{Var}(\mathbb{A}) \) is nontrivial, then any absorption identity that holds in \( \mathbb{B} \) must also hold in \( \mathbb{A} \).

- In the partial semilattice case, there are no absorption identities at all (aside from idempotence).

- The dispersive case can alternatively be described as the case where every absorption identity follows from \((\mathbb{D})_2\):

  \[ \forall n \geq 0 \quad t(...) \approx x. \]

I call it “dispersive” because there is very little absorption.
Absorption identities

- An absorption identity is an identity of the form
  \[ t(x_1, \ldots, x_n) \approx x_i. \]

- If \( A \) is clone-minimal and \( B \in \text{Var}(A) \) is nontrivial, then any absorption identity that holds in \( B \) must also hold in \( A \).

- In the partial semilattice case, there are no absorption identities at all (aside from idempotence).
Absorption identities

- An *absorption identity* is an identity of the form
  \[ t(x_1, \ldots, x_n) \approx x_i. \]

- If \( A \) is clone-minimal and \( B \in \text{Var}(A) \) is nontrivial, then any absorption identity that holds in \( B \) must also hold in \( A \).

- In the partial semilattice case, there are no absorption identities at all (aside from idempotence).

- The dispersive case can alternatively be described as the case where every absorption identity follows from \((D2)\):
  \[ \forall n \geq 0 \quad x(...((xy_1)y_2)\cdots y_n)) \approx x. \]

  I call it “dispersive” because there is very little absorption.
Partial semilattice case

- An idempotent binary operation $s$ is a partial semilattice if

$$s(x, s(x, y)) \approx s(s(x, y), x) \approx s(x, y).$$
Partial semilattice case

- An idempotent binary operation $s$ is a *partial semilattice* if
  
  $$s(x, s(x, y)) \approx s(s(x, y), x) \approx s(x, y).$$

- **Proposition**

  A finite idempotent algebra $\mathbb{A}$ has $a \neq b \in \mathbb{A}$ with
  
  $$(b, b) \in Sg_{\mathbb{A}^2}\{ (a, b), (b, a) \}$$

  if and only if it has a nontrivial partial semilattice operation.
Partial semilattice case

- An idempotent binary operation $s$ is a partial semilattice if
  
  $\text{s}(x, \text{s}(x, y)) \approx \text{s}(\text{s}(x, y), x) \approx \text{s}(x, y)$.

- Proposition

  A finite idempotent algebra $\mathbf{A}$ has $a \neq b \in \mathbf{A}$ with

  $$(b, b) \in \text{Sg}_{\mathbf{A}^2}\{(a, b), (b, a)\}$$

  if and only if it has a nontrivial partial semilattice operation.

- Proof sketch: Let $\text{t}(a, b) = \text{t}(b, a) = b$, then take

  $t^{n+1}(x, y) := t(x, t^n(x, y)),$

  $t^{\infty}(x, y) := \lim_{n \to \infty} t^n(x, y),$ 

  $u(x, y) := t^{\infty}(x, t^{\infty}(y, x)),$ 

  $s(x, y) := u^{\infty}(x, y).$
Rectangular band case

- If $A = (A, f)$ is not Taylor, then $\text{Var}(A)$ must contain a set, on which $f$ either acts as first projection or second projection.
Rectangular band case

- If $A = (A, f)$ is not Taylor, then $\text{Var}(A)$ must contain a set, on which $f$ either acts as first projection or second projection.

- Suppose $B_1, B_2 \in \text{Var}(A)$ are sets such that $f^{B_1} = \pi_1$ and $f^{B_2} = \pi_2$. Let $B = B_1 \times B_2$. 
Rectangular band case

- If $A = (A, f)$ is not Taylor, then $\text{Var}(A)$ must contain a set, on which $f$ either acts as first projection or second projection.

- Suppose $B_1, B_2 \in \text{Var}(A)$ are sets such that $f^{B_1} = \pi_1$ and $f^{B_2} = \pi_2$. Let $B = B_1 \times B_2$.

- The following absorption identities hold on $B$:

  $u \approx f(f(f(u, x), y), f(z, f(w, u)))$,
  $x \approx f(f(x, w), x)$,
  $w \approx f(w, f(x, w))$. 

Rectangular band case

- If $A = (A, f)$ is not Taylor, then $\text{Var}(A)$ must contain a set, on which $f$ either acts as first projection or second projection.

- Suppose $B_1, B_2 \in \text{Var}(A)$ are sets such that $f^{B_1} = \pi_1$ and $f^{B_2} = \pi_2$. Let $B = B_1 \times B_2$.

- The following absorption identities hold on $B$:
  
  $$u \approx f(f(f(u, x), y), f(z, f(w, u))),$$  
  $$x \approx f(f(x, w), x),$$  
  $$w \approx f(w, f(x, w)).$$  

- Take $u = f(x, w)$, get
  
  $$f(f(x, y), f(z, w)) \approx f(x, w),$$

  so $A$ is a rectangular band.
If $\mathbb{A}$ is not a rectangular band, then there is only one type of set in $\text{Var}(\mathbb{A})$, and every binary function restricts to either first or second projection on this set.
If $\mathbb{A}$ is not a rectangular band, then there is only one type of set in $\text{Var}(\mathbb{A})$, and every binary function restricts to either first or second projection on this set.

We define $\text{Clo}_{2}^{\pi_1}(\mathbb{A})$ to be the collection of binary terms of $\mathbb{A}$ which restrict to first projection.
If $\mathbb{A}$ is not a rectangular band, then there is only one type of set in $\text{Var}(\mathbb{A})$, and every binary function restricts to either first or second projection on this set.

We define $\text{Clo}^{\pi_1}(\mathbb{A})$ to be the collection of binary terms of $\mathbb{A}$ which restrict to first projection.

There is a unique surjection from $\mathcal{F}_\mathbb{A}(x, y)$ onto a two-element set, and $\text{Clo}^{\pi_1}(\mathbb{A})$ is one of the congruence classes of the kernel.
\textbf{Clo}_2^{\pi_1}(\mathbb{A})

- If $\mathbb{A}$ is \textit{not} a rectangular band, then there is only one type of set in $\text{Var}(\mathbb{A})$, and every binary function restricts to either first or second projection on this set.

- We define $\text{Clo}_2^{\pi_1}(\mathbb{A})$ to be the collection of binary terms of $\mathbb{A}$ which restrict to first projection.

- There is a unique surjection from $\mathcal{F}_{\mathbb{A}}(x, y)$ onto a two-element set, and $\text{Clo}_2^{\pi_1}(\mathbb{A})$ is one of the congruence classes of the kernel.

- From here on, every function we name will always be assumed to be an element of $\text{Clo}_2^{\pi_1}(\mathbb{A})$. 
Lemma

Suppose $A$ is a binary minimal clone, not Taylor, not a rectangular band, and not a partial semilattice. Then for any $f, g \in \text{Clo}_2(A)$, we have

$$f(x, g(x, y)) \approx x.$$
Crucial lemma

Lemma

Suppose $\mathbb{A}$ is a binary minimal clone, not Taylor, not a rectangular band, and not a partial semilattice. Then for any $f, g \in \text{Clo}_{2}^{\pi_{1}}(\mathbb{A})$, we have

$$f(x, g(x, y)) \approx x.$$  

Proof hints: WLOG every proper subalgebra and quotient of $\mathbb{A}$ is a set.
Crucial lemma

Lemma

Suppose $\mathbb{A}$ is a binary minimal clone, not Taylor, not a rectangular band, and not a partial semilattice. Then for any $f, g \in \text{Clo}_2^{\pi_1}(\mathbb{A})$, we have

$$f(x, g(x, y)) \approx x.$$ 

Proof hints: WLOG every proper subalgebra and quotient of $\mathbb{A}$ is a set.

- If $f(a, g(a, b)) \neq a$, then $a, g(a, b)$ must generate $\mathbb{A}$, so there is $h \in \text{Clo}_2^{\pi_1}(\mathbb{A})$ such that $h(a, b) = b$. 

Lemma

Suppose $\mathbb{A}$ is a binary minimal clone, not Taylor, not a rectangular band, and not a partial semilattice. Then for any $f, g \in \text{Clo}_{2}^{\pi_1}(\mathbb{A})$, we have

$$f(x, g(x, y)) \approx x.$$

Proof hints: WLOG every proper subalgebra and quotient of $\mathbb{A}$ is a set.

- If $f(a, g(a, b)) \neq a$, then $a, g(a, b)$ must generate $\mathbb{A}$, so there is $h \in \text{Clo}_{2}^{\pi_1}(\mathbb{A})$ such that $h(a, b) = b$.
- Consider the relation $Sg_{\mathbb{A}^2}\{(a, b), (b, a)\}$: either it’s the graph of an automorphism, or it has a nontrivial linking congruence, or it’s linked.
Lemma

Suppose \( \mathbb{A} \) is a binary minimal clone, not Taylor, not a rectangular band, and not a partial semilattice. Then for any \( f, g \in \text{Clo}_{2}^{\pi_{1}}(\mathbb{A}) \), we have

\[ f(x, g(x, y)) \approx x. \]

Proof hints: WLOG every proper subalgebra and quotient of \( \mathbb{A} \) is a set.

- If \( f(a, g(a, b)) \neq a \), then \( a, g(a, b) \) must generate \( \mathbb{A} \), so there is \( h \in \text{Clo}_{2}^{\pi_{1}}(\mathbb{A}) \) such that \( h(a, b) = b \).

- Consider the relation \( \text{Sg}_{\mathbb{A}^{2}}\{(a, b), (b, a)\} \): either it’s the graph of an automorphism, or it has a nontrivial linking congruence, or it’s linked.

- If it’s linked, then there is \( \mathbb{B} \subsetneq \mathbb{A} \) such that \( \mathbb{B} \times \mathbb{A} \cap \text{Sg}_{\mathbb{A}^{2}}\{(a, b), (b, a)\} \) is subdirect... from here it’s easy.
Groupy case

- There are three ways to combine binary functions which define associative operations on $\text{Clo}^\pi_2(\mathbb{A})$:
Groupy case

There are three ways to combine binary functions which define associative operations on $\text{Clo}_{2}^{\pi_{1}}(A)$:

- $f, g \mapsto f(x, g(x, y))$,
Groupy case

- There are three ways to combine binary functions which define associative operations on $\text{Clo}_2^{\pi_1}(\mathbb{A})$:
  
  - $f, g \mapsto f(x, g(x, y))$,
  
  - $f, g \mapsto f(g(x, y), y)$,
Groupy case

There are three ways to combine binary functions which define associative operations on $\text{Clo}_{2}^{\pi 1}(A)$:

- $f, g \mapsto f(x, g(x, y))$,
- $f, g \mapsto f(g(x, y), y)$,
- $f, g \mapsto f(g(x, y), g(y, x))$. 
There are three ways to combine binary functions which define associative operations on \( \text{Clo}_{1}^{\pi_{2}}(A) \):

- \( f, g \mapsto f(x, g(x, y)) \),
- \( f, g \mapsto f(g(x, y), y) \),
- \( f, g \mapsto f(g(x, y), g(y, x)) \).

The first one is boring by the Lemma.
There are three ways to combine binary functions which define associative operations on $\text{Clo}_2^{\pi_1}(\mathbb{A})$:

- $f, g \mapsto f(x, g(x, y))$,
- $f, g \mapsto f(g(x, y), y)$,
- $f, g \mapsto f(g(x, y), g(y, x))$.

The first one is boring by the Lemma.

What happens if one of the other two operations forms a group on $\text{Clo}_2^{\pi_1}(\mathbb{A})$?
If the operation $f, g \mapsto f(g(x, y), y)$ forms a group on $\text{Clo}^{\pi_1}_{2}(\mathbb{A})$, then we can use orbit-stabilizer to find nontrivial $f, g \in \text{Clo}^{\pi_1}_{2}(\mathbb{A})$ such that

$$f(x, g(y, x)) \approx f(x, y).$$

Together with the Lemma from before, we see that $f(x, g(y, z)) = f(x, y)$ whenever two of $x, y, z$ are equal.

If $f^{-}$ is the inverse to $f$ in this group, we get $f^{-}(f(x, g(y, z)), y) = x$ whenever two of $x, y, z$ are equal. Semiprojection?
If the operation $f, g \mapsto f(g(x, y), y)$ forms a group on $\text{Clo}_{\pi_1}(\mathbb{A})$, then we can use orbit-stabilizer to find nontrivial $f, g \in \text{Clo}_{\pi_1}(\mathbb{A})$ such that

$$f(x, g(y, x)) \approx f(x, y).$$

Together with the Lemma from before, we see that

$$f(x, g(y, z)) = f(x, y)$$

whenever two of $x, y, z$ are equal.
If the operation \( f, g \mapsto f(g(x, y), y) \) forms a group on \( \text{Clo}_2^{\pi_1}(\mathbb{A}) \), then we can use orbit-stabilizer to find nontrivial \( f, g \in \text{Clo}_2^{\pi_1}(\mathbb{A}) \) such that

\[
f(x, g(y, x)) \approx f(x, y).
\]

Together with the Lemma from before, we see that

\[
f(x, g(y, z)) = f(x, y)
\]

whenever two of \( x, y, z \) are equal.

If \( f^{-} \) is the inverse to \( f \) in this group, we get

\[
f^{-}(f(x, g(y, z)), y) = x
\]

whenever two of \( x, y, z \) are equal. Semiprojection?
Groupy case is $p$-cyclic groupoids

- We have nontrivial $f, g \in \text{Clo}_{2}^{\pi_{1}}(\mathbb{A})$ such that $f(x, g(y, z)) \approx f(x, y)$. 
Groupy case is $p$-cyclic groupoids

- We have nontrivial $f, g \in \text{Clo}^{\pi_1}(\mathbb{A})$ such that $f(x, g(y, z)) \approx f(x, y)$.

- Since $f \in \text{Clo}(g)$, we have

\[ f(x, f(y, z)) \approx f(x, y). \]
Groupy case is $p$-cyclic groupoids

- We have nontrivial $f, g \in \text{Clo}_{\pi}^{1}(A)$ such that $f(x, g(y, z)) \approx f(x, y)$.

- Since $f \in \text{Clo}(g)$, we have

  $$f(x, f(y, z)) \approx f(x, y).$$

- Playing with inverses again, we get

  $$f(f(x, y), x) \approx f(f(x, y), f^{-1}(f(x, y), y)) \approx f(x, y),$$
Groupy case is $p$-cyclic groupoids

- We have nontrivial $f, g \in \text{Clo}_2^{\pi_1}(A)$ such that $f(x, g(y, z)) \approx f(x, y)$.

- Since $f \in \text{Clo}(g)$, we have
  \[ f(x, f(y, z)) \approx f(x, y). \]

- Playing with inverses again, we get
  \[ f(f(x, y), x) \approx f(f(x, y), f^-(f(x, y), y)) \approx f(x, y), \]

- Thus
  \[ f(f(x, y), z) = f(f(x, z), y) \]
  whenever two of $x, y, z$ are equal.
An idempotent groupoid $\mathbb{A}$ is a $p$-cyclic groupoid if it satisfies

\[
x(yz) \approx xy,
\]
\[
(xy)z \approx (xz)y,
\]
\[
(\cdots((xy)y)\cdots y) \approx x,
\]

where the last identity has $p$ ys.
$p$-cyclic groupoids

- An idempotent groupoid $A$ is a $p$-cyclic groupoid if it satisfies

\[
x(yz) \approx xy,
\]
\[
(xy)z \approx (xz)y,
\]
\[
(\cdots ((xy)y) \cdots y) \approx x,
\]

where the last identity has $p$ ys.

- **Theorem (Z.)**

If a binary minimal clone is not a rectangular band and does not have any nontrivial term $f$ satisfying the identity

\[
f(f(x, y), y) \approx f(x, y),
\]

then it is a $p$-cyclic groupoid for some prime $p$. (And similarly if there is no $f(f(x, y), f(y, x)) \approx f(x, y).$)
Structure of \( p \)-cyclic groupoids

- \( p \)-cyclic groupoids were studied by Płonka, who showed they form minimal clones.

The general \( p \)-cyclic groupoid can be written as a disjoint union of affine spaces \( A_1, \ldots, A_n \) over \( \mathbb{F}_p \), together with vectors \( v_{ij} \in A_i \) for all \( i, j \), such that \( x \in A_i, y \in A_j \Rightarrow xy = x + v_{ij} \in A_i \).

The \( v_{ij} \) must satisfy \( v_{ii} = 0 \), and for any fixed \( i \) the set of \( v_{ij} \)s have to span \( A_i \).

The free \( p \)-cyclic groupoid on \( n \) generators has \( np^n - 1 \) elements.
Structure of $p$-cyclic groupoids

- $p$-cyclic groupoids were studied by Płonka, who showed they form minimal clones.

- The general $p$-cyclic groupoid can be written as a disjoint union of affine spaces $A_1, \ldots, A_n$ over $\mathbb{F}_p$, together with vectors $v_{ij} \in A_i$ for all $i, j$, such that

$$x \in A_i, y \in A_j \implies xy = x + v_{ij} \ (\in A_i).$$
Structure of $p$-cyclic groupoids

- $p$-cyclic groupoids were studied by Płonka, who showed they form minimal clones.

- The general $p$-cyclic groupoid can be written as a disjoint union of affine spaces $A_1, \ldots, A_n$ over $\mathbb{F}_p$, together with vectors $v_{ij} \in A_i$ for all $i, j$, such that

  $$x \in A_i, y \in A_j \implies xy = x + v_{ij} \ (\in A_i).$$

- The $v_{ij}$ must satisfy $v_{ii} = 0$, and for any fixed $i$ the set of $v_{ij}$s have to span $A_i$. 
Structure of $p$-cyclic groupoids

- $p$-cyclic groupoids were studied by Płonka, who showed they form minimal clones.

- The general $p$-cyclic groupoid can be written as a disjoint union of affine spaces $A_1, \ldots, A_n$ over $\mathbb{F}_p$, together with vectors $v_{ij} \in A_i$ for all $i, j$, such that

$$x \in A_i, y \in A_j \implies xy = x + v_{ij} \ (\in A_i).$$

- The $v_{ij}$ must satisfy $v_{ii} = 0$, and for any fixed $i$ the set of $v_{ij}$s have to span $A_i$.

- The free $p$-cyclic groupoid on $n$ generators has $np^{n-1}$ elements.
Neighborhood algebras

- An idempotent groupoid is a *neighborhood algebra* if it satisfies the identity

\[(xy)(zx) \approx xy.\]
Neighborhood algebras

▶ An idempotent groupoid is a *neighborhood algebra* if it satisfies the identity

\[(xy)(zx) \approx xy.\]

▶ This is equivalent to satisfying the absorption identity

\[x((yx)z) \approx x.\]
Neighborhood algebras

- An idempotent groupoid is a neighborhood algebra if it satisfies the identity

\[(xy)(zx) \approx xy.\]

- This is equivalent to satisfying the absorption identity

\[x((yx)z) \approx x.\]

- Proposition

*If an idempotent groupoid satisfies \(x(xy) \approx x(yx) \approx x\) and has no ternary semiprojections, then it is a neighborhood algebra.*
Neighborhood algebras

- An idempotent groupoid is a *neighborhood algebra* if it satisfies the identity

\[(xy)(zx) \approx xy.\]

- This is equivalent to satisfying the absorption identity

\[x((yx)z) \approx x.\]

- **Proposition**

  *If an idempotent groupoid satisfies* \(x(xy) \approx x(yx) \approx x\) *and has no ternary semiprojections, then it is a neighborhood algebra.*

- **Proposition (Lévai, Pálfy)**

  *Every neighborhood algebra forms a minimal clone.*
Structure of neighborhood algebras

- In a neighborhood algebra, if $ab = a$ then $ba = b$:

  $$ba = (bb)(ab) = bb = b.$$
Structure of neighborhood algebras

- In a neighborhood algebra, if $ab = a$ then $ba = b$:

$$ba = (bb)(ab) = bb = b.$$  

- Make a graph by drawing an edge connecting $a$ to $b$ whenever $ab = a$.  

- Conversely: Start from any graph such that some vertex is adjacent to all others, and define an idempotent operation by $ab = a$ if $a$, $b$ are connected by an edge, and otherwise let $ab$ be any vertex which is connected to $a$, $b$, and every neighbor of $a$.  

The resulting groupoid will then be a neighborhood algebra.
Structure of neighborhood algebras

- In a neighborhood algebra, if \( ab = a \) then \( ba = b \):

  \[
  ba = (bb)(ab) = bb = b.
  \]

- Make a graph by drawing an edge connecting \( a \) to \( b \) whenever \( ab = a \).

- For any \( a, b \), \( ab \) is connected to \( a, b \), and every neighbor of \( a \).
Structure of neighborhood algebras

- In a neighborhood algebra, if $ab = a$ then $ba = b$:

$$ba = (bb)(ab) = bb = b.$$  

- Make a graph by drawing an edge connecting $a$ to $b$ whenever $ab = a$.

- For any $a, b$, $ab$ is connected to $a$, $b$, and every neighbor of $a$.

- Conversely: Start from any graph such that some vertex is adjacent to all others, and define an idempotent operation by $ab = a$ if $a, b$ are connected by an edge, and otherwise let $ab$ be any vertex which is connected to $a, b$, and every neighbor of $a$. The resulting groupoid will then be a neighborhood algebra.
Structure of neighborhood algebras

- In a neighborhood algebra, if $ab = a$ then $ba = b$:
  \[ ba = (bb)(ab) = bb = b. \]

- Make a graph by drawing an edge connecting $a$ to $b$ whenever $ab = a$.

- For any $a, b$, $ab$ is connected to $a, b$, and every neighbor of $a$.

- Conversely: Start from any graph such that some vertex is adjacent to all others, and define an idempotent operation by $ab = a$ if $a, b$ are connected by an edge, and otherwise let $ab$ be any vertex which is connected to $a, b$, and every neighbor of $a$.

- The resulting groupoid will then be a neighborhood algebra.
Suppose we are not in any of the previous cases.
Dispersive case

- Suppose we are not in any of the previous cases.
- Our crucial Lemma shows that

\[ x \cdots ((xy_1)y_2) \cdots y_n \approx x \]

whenever at most two different variables show up on the left hand side. Semiprojection?
Dispersive case

- Suppose we are not in any of the previous cases.

- Our crucial Lemma shows that

\[ x(\cdots((xy_1)y_2)\cdots y_n) \approx x \]

whenever at most two different variables show up on the left hand side. Semiprojection?

- We need to construct a surjection \( \mathcal{F}_A(x, y) \rightarrow \mathcal{F}_D(x, y) \).
Dispersive case

▶ Suppose we are not in any of the previous cases.

▶ Our crucial Lemma shows that

\[ x(\cdots ((xy_1)y_2)\cdots y_n) \approx x \]

whenever at most two different variables show up on the left hand side. Semiprojection?

▶ We need to construct a surjection \( F_A(x, y) \rightarrow F_D(x, y) \).

▶ The kernel should have equivalence classes \( \{x\}, \{y\}, \text{Clo}^{\pi_1}(A) \setminus \{x\}, \text{and} \text{Clo}^{\pi_2}(A) \setminus \{y\} \).
Suppose, for contradiction, that \( f, g \in \text{Clo}_2^{\pi_1}(A) \) are nontrivial and satisfy

\[
f(x, g(y, x)) \approx x.
\]
Dispersive case - continued

- Suppose, for contradiction, that \( f, g \in \text{Clo}^\pi_2(\mathbb{A}) \) are nontrivial and satisfy
  \[
  f(x, g(y, x)) \approx x.
  \]

- WLOG every proper subalgebra and quotient of \( \mathbb{A} \) is a set (and so \( \text{Clo}^\pi_2(\mathbb{A}) \) is a set).
Suppose, for contradiction, that \( f, g \in \text{Clo}_2^{\pi_1}(\mathbb{A}) \) are nontrivial and satisfy
\[
f(x, g(y, x)) \approx x.
\]
WLOG every proper subalgebra and quotient of \( \mathbb{A} \) is a set (and so \( \text{Clo}_2^{\pi_1}(\mathbb{A}) \) is a set).

For every \( n \), we have
\[
f(x, g(\ldots g(g(y, x), z_1), \ldots, z_n)) \approx x
\]
whenever at most two different variables show up on the left hand side. Semiprojection?
Dispersive case - continued

- Suppose, for contradiction, that $f, g \in \text{Clo}_{\mathbb{A}}^{\pi_1}(A)$ are nontrivial and satisfy
  \[ f(x, g(y, x)) \approx x. \]

- WLOG every proper subalgebra and quotient of $\mathbb{A}$ is a set (and so $\text{Clo}_{\mathbb{A}}^{\pi_1}(A)$ is a set).

- For every $n$, we have
  \[ f(x, g(...g(g(y, x), z_1), ..., z_n)) \approx x \]
  whenever at most two different variables show up on the left hand side. Semiprojection?

- Since we aren’t a neighborhood algebra, there must be some $a, b$ such that
  \[ g(a, g(b, a)) \neq a. \]
Dispersive case - continued

- We have $Sg_\mathbb{A}\{a, g(b, a)\} = \mathbb{A}$ and

  $$f(a, g(...g(g(b, a), z_1), ..., z_n)) \approx a$$

for all $z_1, ..., z_n$. 

By (D2), also have

$$f(a, g(...g(g(b, a), z_1), ..., z_n)) \approx a$$

for all $z_1, ..., z_n$.

Thus, for all $c \in \mathbb{A}$ we have

$$f(a, c) = a.$$ 

Since $g \in Clo(f)$, we get

$$g(a, g(b, a)) = a,$$ 

a contradiction.
Dispersive case - continued

- We have $\text{Sg}_A\{a, g(b, a)\} = A$ and

  $$f(a, g(...g(g(b, a), z_1), ..., z_n)) \approx a$$

  for all $z_1, ..., z_n$.

- By (D2), also have

  $$f(a, g(...g(a, z_1), ..., z_n)) \approx a$$

  for all $z_1, ..., z_n$. 
Dispersive case - continued

- We have \( Sg_\mathbb{A}\{a, g(b, a)\} = \mathbb{A} \) and

\[
f(a, g(...g(g(b, a), z_1), ..., z_n)) \approx a
\]

for all \( z_1, ..., z_n \).

- By \((\mathcal{D}2)\), also have

\[
f(a, g(...g(a, z_1), ..., z_n)) \approx a
\]

for all \( z_1, ..., z_n \).

- Thus, for all \( c \in \mathbb{A} \) we have

\[
f(a, c) = a.
\]
Dispersive case - continued

- We have $Sg_{\mathbb{A}}\{a, g(b, a)\} = \mathbb{A}$ and

$$f(a, g(...g(g(b, a), z_1), ..., z_n)) \approx a$$

for all $z_1, ..., z_n$.

- By ($D2$), also have

$$f(a, g(...g(a, z_1), ..., z_n)) \approx a$$

for all $z_1, ..., z_n$.

- Thus, for all $c \in \mathbb{A}$ we have

$$f(a, c) = a.$$ 

- Since $g \in \text{Clo}(f)$, we get $g(a, g(b, a)) = a$, a contradiction.
Need to rule out two similar possibilities - the arguments are similar, but now we must use the existence of functions satisfying $f(f(x, y), y) \approx f(x, y)$ or $f(f(x, y), f(y, x)) \approx f(x, y)$. 

I don't know if this is true: Conjecture If $A$ is a dispersive binary minimal clone, then for any $a \neq b$ there is a surjective map from $S_g A \{a, b\}$ to a two-element set.
Need to rule out two similar possibilities - the arguments are similar, but now we must use the existence of functions satisfying $f(f(x, y), y) \approx f(x, y)$ or $f(f(x, y), f(y, x)) \approx f(x, y)$.

To see that $Sg_{A^2}\{(a, b), (b, a)\} \rightarrow F_D(x, y)$ when $\{a, b\}$ is not a subalgebra, note that if $f((a, b), (b, a)) = (a, b)$, then we must have $f(x, y) \approx x$. 

I don't know if this is true: Conjecture If $A$ is a dispersive binary minimal clone, then for any $a \neq b$ there is a surjective map from $Sg_{A^2}\{(a, b), (b, a)\}$ to a two-element set.
Need to rule out two similar possibilities - the arguments are similar, but now we must use the existence of functions satisfying \( f(f(x, y), y) \approx f(x, y) \) or \( f(f(x, y), f(y, x)) \approx f(x, y) \).

To see that \( Sg_{\mathbb{A}^2}\{(a, b), (b, a)\} \rightarrow \mathcal{F}_D(x, y) \) when \( \{a, b\} \) is not a subalgebra, note that if \( f((a, b), (b, a)) = (a, b) \), then we must have \( f(x, y) \approx x \).

I don’t know if this is true:

**Conjecture**

If \( \mathbb{A} \) is a dispersive binary minimal clone, then for any \( a \neq b \) there is a surjective map from \( Sg_{\mathbb{A}}\{a, b\} \) to a two-element set.
Thank you for your attention.