Interpolating log

Sometimes we wish to find a function perfectly in between \( x \) and \( e^x \). That is, we desire a function \( f \) such that \( f(f(x)) = e^x \), at least asymptotically. There are slight technical difficulties with finding a function which exactly satisfies \( f(f(x)) = e^x \), but it turns out that we can find a nice bijective function \( f : [0, \infty) \to [0, \infty) \) which satisfies

\[
  f(f(x)) = e^x - 1.
\]

The advantage of using \( e^x - 1 \) here is that \( e^0 - 1 = 0 \), so we can set \( f(0) = 0 \).

We define a pair of functions \( \varepsilon(x) \) and \( \ell(x) \) by

\[
  \varepsilon(x) = e^x - 1
\]

and

\[
  \ell(x) = \ln(1 + x),
\]

and note that \( \varepsilon, \ell : [0, \infty) \to [0, \infty) \) are inverse bijections.

For each \( n \in \mathbb{N} \), we define \( \varepsilon^n(x) \) and \( \ell^n(x) \) to be the \( n \)th iterates of \( \varepsilon \) and \( \ell \), so that \( \varepsilon^0(x) = \ell^0(x) = x \) and \( \varepsilon^{n+1}(x) = \varepsilon(\varepsilon^n(x)), \ell^{n+1}(x) = \ell(\ell^n(x)) \). The strategy is to start by defining a bijective function \( \ell^* : (0, \infty) \to \mathbb{R} \) such that \( \ell^*(1) = 0 \),

\[
  \ell^*(\varepsilon(x)) = \ell^*(x) + 1,
\]

and

\[
  \ell^*(\ell(x)) = \ell^*(x) - 1.
\]

Intuitively, \( \ell^*(x) \) is “the number of times we have to apply \( \ell \) to reach 1”. Using \( \ell^* \), we can then construct a function \( \varepsilon^{1/2} \) which satisfies \( \varepsilon^{1/2}(\varepsilon^{1/2}(x)) = e^x - 1 \).

**Proposition 1.** For all \( x > 0 \), we have \( \varepsilon(x) > x \) and \( \ell(x) < x \). In particular, for any \( x > 0 \), we have \( \lim_{n \to \infty} \ell^n(x) = 0 \).

The intuition for computing \( \ell^*(x) \) is that we may use the identity

\[
  \ell^*(\ell^n(x)) = \ell^*(x) - n
\]

to reduce the computation of \( \ell^*(x) \) to the computation of \( \ell^*(\ell^n(x)) \). Since \( \ell^n(x) \) is eventually quite close to 0, we just need to understand how \( \ell \) acts on numbers close to 0. We can approximate \( \ell(x) \) for small \( x \) by the Taylor series

\[
  \ell(x) = x - \frac{x^2}{2} + O(x^3).
\]

Comparing \( \frac{1}{\ell(x)} \) to \( \frac{1}{x} \), we get the following estimate.
Proposition 2. For $x$ small, we have

$$\frac{1}{\ell(x)} = \frac{1}{x} + \frac{1}{2} - \frac{x}{12} + O(x^2).$$

Additionally, we have

$$\frac{1}{2} - \frac{x}{12} \leq \frac{1}{\ell(x)} - \frac{1}{x} < \frac{1}{2}$$

for all $x > 0$.

Proof. The first statement follows from standard power series manipulation:

$$\frac{1}{x - x^2/2 + x^3/3 - x^4/4 + x^5/5 - \cdots} = \frac{1}{x} + \frac{1}{2} - \frac{x}{12} + \frac{x^2}{24} - \frac{19x^3}{720} + \cdots.$$

The inequality $\frac{1}{\ell(x)} - \frac{1}{x} < \frac{1}{2}$ is equivalent to

$$\ell(x) > \frac{1}{1/x + 1/2} = 2 - \frac{4}{2+x},$$

and since this is true for $x$ sufficiently close to 0, we just need to check that the derivative of the left hand side is at least the derivative of the right hand side. Thus we just need to check that

$$\frac{1}{1 + x} > \frac{4}{(2 + x)^2},$$

which follows by multiplying out.

We only need to check the inequality $\frac{1}{2} - \frac{x}{12} < \frac{1}{\ell(x)} - \frac{1}{x}$ in the range $0 < x < 6$, and in this range it is equivalent to

$$\ell(x) < \frac{1}{1/x + 1/2 - x/12} = \frac{x}{1 + x/2 - x^2/12}.$$

Again, this is true for $x$ sufficiently close to 0, so we may compare the derivatives instead. We see that we just need to check that

$$\frac{1}{1 + x} < \frac{(1 + x/2 - x^2/12) - x(1/2 - x/6)}{(1 + x/2 - x^2/12)^2} = \frac{1 + x^2/12}{(1 + x/2 - x^2/12)^2}$$

for $0 < x < 6$. Multiplying out, this becomes

$$(1 + x/2 - x^2/12)^2 < (1 + x)(1 + x^2/12),$$

or

$$1 + x + \frac{x^2}{12} - \frac{x^3}{12} + \frac{x^4}{144} < 1 + x + \frac{x^2}{12} + \frac{x^3}{12},$$

which holds for $0 < x < 24$. \(\square\)

Corollary 1. For $x \leq 1$, we have

$$\frac{5n}{12} < \frac{1}{\ell^n(x)} - \frac{1}{x} < \frac{n}{2}.$$
Corollary 2. For $x \leq 1$, we have
\[ \frac{1}{\ell^n(x)} = \frac{1}{x} + \frac{n}{2} - \sum_{i<n} \frac{\ell^i(x)}{12} + O(x). \]

Corollary 3. For $x \leq 1$, we have
\[ \frac{1}{\ell^n(x)} = \frac{1}{x} + \frac{n}{2} - O(\ln(n)). \]

Corollary 4. For $x$ fixed and $n$ going to infinity, we have
\[ \frac{1}{\ell^n(x)} = \frac{n}{2} - \frac{\ln(n)}{6} + O_x(1). \]

So one natural path to computing $\ell^*(x)$ is to try to compute
\[ \lim_{n \to \infty} n - \frac{\ln(n)}{3} - \frac{2}{\ell^n(x)}. \]

A simpler approach is to compare $\frac{2}{\ell^n(x)}$ to $\frac{2}{\ell^n(1)}$.

Proposition 3. For any $x, y > 0$, we have
\[ \left| \frac{1}{x} - \frac{1}{y} \right| \leq \left| \frac{1}{\ell(x)} - \frac{1}{\ell(y)} \right| \leq \left| \frac{1}{x} - \frac{1}{y} \right| + \frac{|x - y|}{12}. \]

Proof. We just need to show that the function \( f(x) = -1/\ell(x) \) has derivative bounded below by \( \frac{1}{x^2} \) and above by \( \frac{1}{x^2} + \frac{1}{12} \). We have
\[ f'(x) = \frac{1}{1 + x} \cdot \frac{1}{\ell(x)^2}. \]
Thus, for the left hand inequality, we just need to check that
\[ \ell(x)^2 < \frac{x^2}{1 + x}, \]
or equivalently
\[ \ell(x) < \frac{x}{(1 + x)^{1/2}}. \]
Since equality holds at 0, it’s enough to compare the derivatives: we just need to show that
\[ \frac{1}{1 + x} < \frac{1}{(1 + x)^{1/2}} - \frac{x}{2(1 + x)^{3/2}}. \]
Multiplying out, this becomes
\[ 2\sqrt{1 + x} < 2 + x, \]
and squaring both sides shows that this holds for all $x > 0$. 3
For the right hand inequality, we need to check that
\[ \ell(x)^2 > \frac{x^2}{(1 + x)(1 + x^2/12)}, \]
or equivalently that
\[ \ell(x) > \frac{x}{(1 + x)^{1/2}(1 + x^2/12)^{1/2}}. \]
Again, it’s enough to compare the derivatives, so we just need to check that
\[
\frac{1}{1 + x} > \frac{1}{(1 + x)^{1/2}(1 + x^2/12)^{1/2}} - \frac{x}{2(1 + x)^{3/2}(1 + x^2/12)^{1/2}} - \frac{x^2}{12(1 + x)^{1/2}(1 + x^2/12)^{3/2}}.
\]
Multiplying out, this becomes
\[(1 + x)^{1/2}(1 + x^2/12)^{3/2} > 1 + x/2 - x^3/24, \]
and on squaring both sides we get the inequality
\[(1 + x)(1 + x^2/12)^{3} > 1 + x + x^2/4 - x^3/12 - x^4/24 + x^6/24^2, \]
which the reader may verify by using the inequality \(x^5 + x^7 \geq 2x^6\).

\[ \square \]

**Corollary 5.** For any \(x, y > 0\), the limit
\[
\lim_{n \to \infty} \frac{2}{\ell^n(y)} - \frac{2}{\ell^n(x)}
\]
exists, and is equal to
\[
\lim_{n \to \infty} \frac{n^2}{2} \left( \ell^n(x) - \ell^n(y) \right).
\]

**Proof.** To see that the limit exists, note that if \(x \geq y\), then the sequence
\[
\frac{2}{\ell^n(y)} - \frac{2}{\ell^n(x)}
\]
is increasing in \(n\) and is bounded above by
\[
\frac{2}{y} - \frac{2}{x} + \sum_{m \geq 0} \frac{\ell^m(x) - \ell^m(y)}{6} \leq \frac{2}{y} - \frac{2}{x} + \frac{kx}{6},
\]
where \(k\) is any integer which satisfies \(y \geq \ell^k(x)\).

For the second statement, note that
\[
\frac{2}{\ell^n(y)} - \frac{2}{\ell^n(x)} = \frac{2(\ell^n(x) - \ell^n(y))}{\ell^n(x)\ell^n(y)},
\]
and use the asymptotic
\[ \ell^n(x) = (1 + o_x(1)) \frac{2}{n}, \]
(and similarly for \(y\)) to replace the denominator by \(4/n^2\).

\[ \square \]
Definition 1. For $x > 0$, we define $\ell^*(x)$ by

$$
\ell^*(x) = \lim_{n \to \infty} \frac{2}{\ell^n(1)} - \frac{2}{\ell^n(x)} = \lim_{n \to \infty} \frac{n^2}{2} \left( \ell^n(x) - \ell^n(1) \right).
$$

Proposition 4. For all $x > 0$, the function $\ell^*(x)$ satisfies

$$
\ell^*(e^x - 1) = \ell^*(x) + 1
$$

and

$$
\ell^*(\ln(1 + x)) = \ell^*(x) - 1.
$$

Proof. It’s enough to prove the second statement. By the definition of $\ell^*$, we have

$$
\ell^*(x) - \ell^*(\ell(x)) = \lim_{n \to \infty} \frac{2}{\ell^{n+1}(x)} - \frac{2}{\ell^n(x)}.
$$

Setting $y_n = \ell^n(x)$, we have $y_n \to 0$, so the above is equal to

$$
\lim_{y \to 0} \frac{2}{\ell(y)} - \frac{2}{y} = 1.
$$

For the sake of concretely approximating $\ell^*$, we have the following explicit bound.

Proposition 5. If $x \geq y \geq \ell^k(x)$, then for any $n$ we have

$$
\frac{2}{\ell^n(y)} - \frac{2}{\ell^n(x)} \leq \ell^*(x) - \ell^*(y) \leq \frac{2}{\ell^n(y)} - \frac{2}{\ell^n(x)} + \frac{k\ell^n(x)}{6}.
$$

Of course, we’d like to know if the function $\ell^*$ is well-behaved: is it continuous, is it differentiable, etc. To answer this question, we use the theory of completely monotone/Bernstein functions.

Definition 2. A continuous function $f : [0, \infty) \to \mathbb{R}$ is called completely monotone if it satisfies

$$
(-1)^n f^{(n)}(x) \geq 0
$$

for all $x > 0$ and all $n \in \mathbb{N}$.

A function $g : [0, \infty) \to [0, \infty)$ whose derivative is completely monotone is called a Bernstein function.

Proposition 6. If $f, g$ are Bernstein, then the composition $f \circ g$ is also a Bernstein function. If $f$ is completely monotone and $g$ is Bernstein, then $f \circ g$ is completely monotone.

Corollary 6. For every $n$, the function $\ell^n$ is a Bernstein function, and $1/\ell^n$ is a completely monotone function.

Proposition 7. If $f$ is a pointwise limit of functions $f_i$ such that for each $n \geq 1$, the derivatives $f_i^{(n)}$ exist and have a fixed sign $s_n \in \{+, -\}$, then each derivative $f^{(n)}$ exists and has the same fixed sign $s_n$. In particular, any pointwise limit of Bernstein functions is a Bernstein function, and the same holds for completely monotone functions.
Proposition 8. Every completely monotone function \( f : (0, \infty) \rightarrow \mathbb{R} \) extends to an analytic function on the halfplane \( \Re(x) > 0 \), as does any Bernstein function.

Corollary 7. The function \( \ell^* \) has completely monotone derivative, and extends to an analytic function on the halfplane \( \Re(x) > 0 \).

We can also define a tetration function \( \varepsilon^* \).

Definition 3. We define \( \varepsilon^* : \mathbb{R} \rightarrow (0, \infty) \) to be the inverse function to \( \ell^* \).

Now we can finally define the fractional compositional powers of the function \( e^x - 1 \).

Definition 4. For every \( n \in \mathbb{R} \), we define the function \( \varepsilon^n : (0, \infty) \rightarrow (0, \infty) \) by
\[
\varepsilon^n(x) = \varepsilon^*(\ell^*(x) + n).
\]
We define \( \ell^n \) by \( \ell^n(x) = \varepsilon^{-n}(x) \).

Proposition 9. For any \( m,n \in \mathbb{R} \) and any \( x > 0 \), we have
\[
\varepsilon^m(\varepsilon^n(x)) = \varepsilon^{m+n}(x).
\]
In particular, we have
\[
\varepsilon^{1/2}(\varepsilon^{1/2}(x)) = e^x - 1.
\]

We can also define an asymptotic measurement of “how exponentially” a function grows.

Definition 5. We say that a function \( f : (0, \infty) \rightarrow (0, \infty) \) has exponentiality \( \alpha(f) \) if
\[
\alpha(f) = \lim_{x \rightarrow \infty} \ell^*(f(x)) - \ell^*(x) = \lim_{x \rightarrow \infty} \ell^*(f(\varepsilon^*(x))) - x.
\]
Under this definition, we have \( \alpha(1) = -\infty \), \( \alpha(x) = 0 \), \( \alpha(\varepsilon) = 1 \), and \( \alpha(\ell) = -1 \). Additionally, we have \( \alpha(\varepsilon^n) = n \) for all \( n \in \mathbb{R} \), \( \alpha(\ell^*) = -\infty \), and \( \alpha(\varepsilon^*) = +\infty \).

Proposition 10. If \( f,g : (0, \infty) \rightarrow [\epsilon, \infty) \) are functions with exponentialities \( \alpha(f), \alpha(g) \), then
\[
\alpha(fg) = \alpha(f + g) = \max(\alpha(f),\alpha(g)).
\]

Proposition 11. If \( f,g : (0, \infty) \rightarrow (0, \infty) \) have exponentialities \( \alpha(f), \alpha(g) > -\infty \), then
\[
\alpha(f \circ g) = \alpha(f) + \alpha(g).
\]

Proof. For any \( x \), we have
\[
\ell^*(f(g(x))) - \ell^*(x) = \ell^*(f(g(x))) - \ell^*(g(x)) + \ell^*(g(x)) - \ell^*(x).
\]
Since \( g(x) \) must go to \( \infty \) as \( x \rightarrow \infty \) if \( \alpha(g) > -\infty \), we see that the limit of the above expression is \( \alpha(f) + \alpha(g) \).

Corollary 8. Every function which can be constructed (in finitely many steps) out of positive polynomials by addition, multiplication, exponentiation, and taking logarithms has an exponentiality in \( \mathbb{Z} \cup \{-\infty\} \).