More Combinatorial bijections.

- Recall \( \binom{n+k-1}{k} = \# \) of \( k \)-multisets of \([n]\) i.e. \( k \) element sets with repeated entries such as \([1,1,2,4,5,5]\) is a \( 6 \)-multiset of \([5]\).

- Notation \( \binom{n+k-1}{k} = \binom{n}{k} \).

\[
\binom{n}{k-m} = \binom{k-1}{n-m} = \binom{k-1}{m-1}.
\]
Is there a formula for partial sums like

\[ \sum_{k=0}^{m} \binom{n}{k} = ? \]

If \( m = n \) then we get \( 2^n \) (Binomial Thm).

Not a 'simple' formula for general \( m, n \).

But,

\[ \sum_{k=0}^{m} \binom{n}{k} = \binom{n+1}{m} \]

Proof: RHS = \# of \( m \)-multisets of \( \{n+1\} \).

(CS: condition on the number of times elements from \( \{n\} \) is chosen.)
If \( k \) total elements from \( \mathbb{C}_n \) is chosen (including multiplicity) then there are \( \binom{n}{k} \) of those.

The remaining \( m-k \) elements will be \( \text{"anti"} \).

So \( \sum_{k=0}^{m} \binom{n}{k} \) in total.

Inclusion-Exclusion with multiplicity

Problem: Given subsets \( A_1, ..., A_n \subseteq U \), let

\[ N_m = \# \text{ of } x \in U \text{ s.t. } x \text{ appears in at least } m \text{ of the sets } A_1, ..., A_n. \]

\[ N_i = \# \bigcup_{i \in I} A_i. \]
Then $x$ contributes $0$ to the RHS of (9).

Hope to prove this in a combinatorial way.

Suppose $x\in A_i$ does not belong to $N_m$ and also

apart of $\bigcup_{1 \leq k \leq m} N_k^{(i)}$.

\[ N_m = \bigcap_{1 \leq k \leq m} N_k^{(i)} \]

Let $A = \{ A_i : i \in \mathbb{N} \}$ for $\mathbb{N} \in \mathbb{N}$. \[ A = \mathbb{N} \]
Suppose \( x \in \mathbb{U} \) belongs to the sets \( A_j \) for \( j \in T \) with \( |T| = m+j \) (Ts \( m+j \)).

Then \( x \) contributes once to \( \sum_{m+j} \) and contributes

\[
\leq \binom{k-1}{m-1} (-1)^{k-m} \binom{m+j}{k}
\]

to the RHS.

Have to show that

\[
\sum_{k=m}^{m+j} \binom{k-1}{m-1} (-1)^{k-m} \binom{m+j}{k} = 1 \quad \text{for} \quad j \geq 0.
\]

0. \( \binom{k-1}{m-1} = \frac{(m)}{k-m} \) \( \text{Reindex} \) \( k \leq k+m \) \( \text{get} \)
\[ \sum_{k=0}^{j} \binom{m}{k} (-1)^{k} \binom{m+j}{j-k} = 1 \quad \text{for } j \geq 0. \]

Consider all pairs \((S, T)\) with \(S, T \subseteq \{m+j\}\) such that:

1. \(S \subseteq \{m\}\) and is a multiset.
2. \(T \subseteq \{m+j\}\) and is a subset.
3. \(|S| + |T| = j\).

\(E\) = all such pairs \((S, T)\) with \(|S|\) even.

\(O\) = \(\ldots\) with \(|S|\) odd.
$$|E| \leq \binom{m}{k} \binom{m+j}{j-k}$$

$k$ even

$$|E| \leq \binom{m}{k} \binom{m+j}{j-k}$$

$k$ odd

Want $|E|-|O|$

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largest $i$. If $i \in T$ then send $i$ to $S$ to

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get $(S \cup i \setminus S, T \setminus i)$. If $i \in T$ then $i \in S$

so send $i$ to $T$ to get $(S \setminus i \cup S, T \cup i)$. 

so send $i$ to $T$ to get $(S \setminus i \cup S, T \cup i)$. 

This map is an involution so long as SUT contains at least one element from \([m]\).

Map sends \(E \cup \{S\} \rightarrow 0 \cup \{S\}
\)

\(0 \cup \{\text{constraint}\} \rightarrow E \cup \{\text{constraint}\}\)

How many pairs \((S,T)\) do not contain at least

1 element from \([m]\).

Only 1 : \(S = \emptyset\) and \(T = \{m+1, \ldots, m+j\}\).

Only \((\emptyset, \{m+1, \ldots, m+j\}) \in E\) and so \(|E| = |0| + 1\).

Thus, \(|E| - |0| = 1\), as needed.
Show that \[ \sum_{k=0}^{n} \binom{n}{k} \binom{k}{m} (-1)^k = (-1)^n \delta_{m,n} \]

where \( \delta_{m,n} = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{otherwise} \end{cases} \).

- Consider pairs \((S,T)\) such that \(S, T \subseteq [n]\) with 
  \(T \subseteq S\) and \(|T| = m\).

\[ E = \{ \text{all pairs with } |S| \text{ even} \} \]

\[ O = \{ \text{all pairs with } |S| \text{ odd} \} \]

We have \(|E| - |O| = \text{LHS of the identity.}\)
If \( m > n \) then there are no pairs.

If \( m = n \) then there is only one term, equals \((-)^n\).

Case \( m < n \).

Given \((s, t) \in x \in \mathbb{N} \mid t\) because \( t < n \).

Pick largest \( x \in \mathbb{N} \mid t\).

If \( x \in S \) then map \((s, t) \rightarrow (s \times x, t)\).

If \( x \notin S \) then map \((s, t) \rightarrow (s \times x^2, t)\).
This map is an involution from all pairs \((s,T)\) into themselves and it maps \(E \to O\) and \(O \to E\).

This is a bijection between \(E\) and \(O \Rightarrow\)

\(|E| = |O| \) as required.