Permutations: The number of ways to arrange $n$ distinct objects is

$$n! = n(n-1)(n-2) \cdots 2 \cdot 1$$

- $n$ choices for the first place, $(n-1)$ choices for the second, $(n-2)$ choices for the third, and so on.
- # of ways = product of all num. of choices.

$$n \ n! \ n-2 \ n-3 \ \cdots$$

- Permutations are also functions $\sigma: \{n\} \rightarrow \{n\}$ that are bijective (one to one and onto).
Combinations:

- How many ways are there to make an ordered list of \( k \) distinct elements from \( n \) elements?

\[ h, (n-1), (n-2) \ldots (n-k+1) =: (n)_k = \binom{n}{k} \]

\( \Rightarrow \) 2nd choices \( \Rightarrow (n-1)_k \) ways to make the choice

1st choice for position 1

- How many ways are there to make an unordered choice of \( k \) distinct elements from \( \binom{n}{k} \)?

\( \# \) of ordered choices = \( (\# \) of unordered choices) \( \times \) \( k! \)

Thus, \( \# \) of unordered choices = \( \frac{(n)_k}{k!} \) ways to order the list.
Binomial Coefficients

\[ \binom{n}{k} = \frac{n!}{k!(n-k)!} \]

is called a binomial coefficient. Need \(0 \leq k \leq n\), but set \(\binom{n}{k} = 0\) if \(k > n\).

\(\binom{n}{k}\) = \# of \(k\)-element subsets of \([n]\) as unordered list are the same as subsets list are the same as subsets list are the same as subsets list are the same as subsets list are the same as subsets.

Note that \(\binom{n}{k}\) can be defined even if \(n\) is not a integer. \(\binom{z}{k} = \frac{z(z-1)\ldots(z-k+1)}{k!} \) for \(z \in \mathbb{C}\) and \(k \geq 0\) integer.

Set \(\binom{z}{k} = \frac{z(z-1)\ldots(z-k+1)}{k!} \) for \(z \in \mathbb{C}\) and \(k \geq 0\) integer.

This is called analytic continuation.
Pascal's Triangle
Binomial coefficients can be arranged in triangular form. How N'th triangle contains the numbers \( \binom{n}{k} \) where \( 0 \leq k \leq n \). Many patterns!
Combinatorial identities

1) \( \binom{n}{k} = \binom{n}{n-k} \)

Proof: k-subsets of \([n]\) are in bijection with \((n-k)\)-subsets via complementation: \( S \) is a k-subset \( \iff \) \([n] \setminus S \) is a \((n-k)\)-subset.

2) \( \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \)

LHS = k-subsets of \([n]\).

RHS counts k-subsets by conditioning on whether \( n \) is included or not. If \( n \) is included then we must select a \((k-1)\)-subset from \([n-1]\) in \( \binom{n-1}{k-1} \) ways. Otherwise, select k-subset from \([n-1]\).
\[ 2) \binom{n}{0} + \binom{n+1}{1} + \ldots + \binom{n+k}{k} = \binom{n+k+1}{k} \quad \text{[hockey stick pattern]} \]

RHS: \# of ways to choose \(n+1\) people from a group of \(n+k+1\). 
\[ \binom{k}{k} = \binom{n+1}{n} \cdot \binom{h+k+1}{h} = \binom{n+k+1}{n+1} . \]

LHS: Condition on the label of the highest person picked. Suppose it is \(n+i\) for some \(k\).
Then there are \(n\) more people to be picked from \(n+i\) people with smaller label.
\# of ways = \(\binom{n+i}{n}\). These are mutually exclusive.
Total \# of ways = \(\sum_{i=0}^{k} \binom{n+i}{n} = \sum_{i=0}^{k} \binom{n+i}{i} \).
\[
(\binom{n}{0})^2 + (\binom{n}{1})^2 + \cdots + (\binom{n}{n})^2 = \binom{2n}{n}
\]

**RHS:** \# of ways to pick \( n \) people from \( 2n \) people.

**LHS:** Partition the \( 2n \) people into 2 groups of

\[n \in \{1, 2, \ldots, n\} \text{ and } \{n+1, \ldots, 2n\}\]

Condition on the \# of people chosen from 1st group.

Suppose it is \( k \) with \( 0 \leq k \leq n \).

Then \( \binom{k}{k} \) ways to pick \( k \) people from 1st group.

Then \( \binom{n}{k-k} = \binom{k}{k} \) ways to pick remaining people from 2nd group.

Total = \( \sum_{k=0}^{n} \binom{k}{k}^2 \), as needed.
Multinomial coefficients

How many ways are there to arrange \( n \) distinct balls into \( r \) distinct boxes where box \( K \) has \( q_k \) balls?

\[
\frac{n!}{q_1!q_2!\cdots q_r!}
\]

\( q_1 + q_2 + \cdots + q_r = n \)

(\( \binom{n}{q_1} \)) ways to put \( q_1 \) balls in box 1

(\( \binom{n-a_1}{q_2} \)) ways to put \( q_2 \) balls in box 2 after insertion into box 1.

(\( \binom{n-a_1}{a_r} \)) ways to put \( a_r \) balls in box \( r \).

\( \cdots \binom{n-a_1-a_2-\cdots-a_{r-1}}{a_r} \) ways to put \( a_r \) balls in box \( r \).

\# of ways = \( \frac{n!}{q_1!q_2!\cdots q_r!} \)
Simplification gives
\[
\binom{n}{a_1 a_2 \ldots a_r} \frac{n!}{a_1! a_2! \cdots a_r!} = \frac{n!}{a_1! a_2! \cdots a_r!}
\]

\[
\frac{n!}{a_1! a_2! \cdots a_r!} = \binom{n}{a_1, \ldots, a_r} \text{ is the multinomial coefficient.}
\]
Binomial Thm:

\[(X+Y)^n = \sum_{k=0}^{n} \binom{n}{k} X^k Y^{n-k} \quad \forall x, y \in \mathbb{C}.
\]

Reality: \(x, y\) are commuting variables: \(xy = yx\).

Pr: The term \(X^k Y^{n-k}\) occurs by choosing \(k\) \(X\)'s from the \(n\)-fold product. There are \(\binom{n}{k}\) ways to choose which of the \(n\) terms will contribute the \(k\) \(X\)'s.
Alternate proof:

\[(1+x)^n = \# \text{ of length } n \text{ sequences where each element of the sequence is a number from } 1, \ldots, 1+x \quad (\text{assume } x \geq 0 \text{ integer})\]

Condition on the \# of occurrences of \(1+x\) in the sequence. Suppose it equals \(n-k\) for some \(k \leq n\).

- Choose the places for \((1+x)\) in \(\binom{n}{n-k} = \binom{n}{k}\) ways.
- Choose the places for \(1, \ldots, x\). There are \(x^k\) of those numbers.

The remaining sequence has length \(k\) and consists of \(1\)'s.

Thus, \(\sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n\).
Now, \((x^n)^k = \sum_{k=0}^{n} \binom{n}{k} x^k\) is a polynomial identity of degree \(n\), true for all \(x > 0\) and integer. It follows that this holds for all \(x > 0\).

\[ X = \frac{a}{b} \quad \text{and multiply by } b^n, \text{ we get} \]

\[ (bta)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}, \quad a, b > 0. \]
Multinomial Thm.: 

\[(X_1 + X_2 + \cdots + X_r)^n \leq \binom{n}{a_1, a_2, \ldots, a_r} X_1^{a_1} X_2^{a_2} \cdots X_r^{a_r},\]

where \(a_1 + a_2 + \cdots + a_r = n\) and \(a_i \geq 0\).

Proof: Thus as before. Use the combinatorial meaning of multinomial coefficient.

Ex.: Do it using the alternate proof idea.