Lecture 1

Pigeon hole principle (PHP)

What is this course about?

2/3 of the course is counting

- direct counting
- combinatorial bijections
- recursion
- generating functions

1/3 of course -> Graph theory
Pigeon hole principle

If you have \( p \) pigeons to be distributed into \( h \) holes then some hole contains at least

\[
\left\lfloor \frac{p}{h} \right\rfloor
\]

pigeons.
Proof of PHP

Suppose this is not the case and every hole has \( \lfloor \frac{p}{h} \rfloor - 1 \) pigeons.

\[
\# \text{ pigeons} \leq h \left( \left\lfloor \frac{p}{h} \right\rfloor - 1 \right)
\]

\[
\left\lfloor \frac{p}{h} \right\rfloor < \frac{p}{h} + 1 < \text{amount you round up by is is}
\]

\[
\frac{p}{h} \quad \text{less than 1}.
\]

\[
\# \text{ pigeons} \leq h \left( \left\lfloor \frac{p}{h} \right\rfloor - 1 \right) < h \left( \frac{p}{h} \right) = p.
\]

Contradiction.
PHP in geometry

You have 9 points distributed inside the unit square. Show that some 3 points form a triangle with area at most $1/9$.

Area of each region $= \frac{1}{2} \left( \frac{1}{3} \right)^2 = \frac{1}{9}$

By the PHP, some region contains at least 3 points!
PHP in number theory

You have numbers \( \{1, 2, \ldots, 2n\} \)

Pick any \( n+1 \) of these \( 2n \) numbers.
Show there are two chosen numbers such one divides the other.

Write every \( m = 2 \cdot \frac{a}{\text{odd part}} \) \( \geq 1 \)

How many possible \( k \)'s for \( 1 \leq m \leq 2n \) ?

\( n \) choices for \( k \) (holes)
By the PHP some two of the \( n+1 \) chosen numbers have the same odd part.

Suppose we have \( m_1 = 2^{a_1} (2k+1) \)
\[m_2 = 2^{a_2} (2k+1)\]

Then \( m_1 \mid m_2 \) if \( a_1 < a_2 \) or vice versa.

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**PHP in combinatorics**

**Ramsey theory**
claim: There is always a monochromatic $\Delta$ in any red/blue edge colouring.
Thm \([\text{Ramsey}]\) If you colour every pair of points \((i,j)\) for \(1 \leq i,j \leq n\) using red or blue then the following holds. For every \(k\) there is a \(N\) such that if \(n \geq N\) then some set of \(k\) points have monochromatic edges.

\[
R(k,k) = \text{minimal value of } N.
\]

\[
R(3,3) \leq 6; \quad \text{Exercise } R(3,3) = 6.
\]
Thm [Erdős]: $R(k,k) \geq 2^{k/2}$. 

Open: $R(k,k) \geq (2+\epsilon)^{k/2}$ for some $\epsilon > 0$?

PHP in Analysis

Thm [Dirichlet]: Given any irrational number $\alpha$ and $\epsilon > 0$, there is a rational number $\frac{p}{q}$ with $1 \leq q \leq \frac{1}{\epsilon}$ such that

$$|\alpha - \frac{p}{q}| \leq \frac{\epsilon}{q^2} \leq \frac{1}{q^2}.$$

Integer multiples of $\alpha$ can be arbitrarily close to integers!
Proof: Write real number \( x = \lfloor x \rfloor + \{ x \} \) fractional part closest integer odd down.

\( (x) \in \mathbb{Z}, \ (a) \in [0,1) \): 

Holes: 

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\( \varepsilon \) subdivide \([0,1)\) into intervals of length \( \leq \varepsilon\) with \( \leq \lceil \frac{1}{\varepsilon} \rceil + 1\) subintervals.

Pigeons: \( \delta, 2\delta, 3\delta, \ldots, (\lceil \frac{1}{\varepsilon} \rceil + 1) \delta \)

By PHP there are \( a \leq a < b \leq \lceil \frac{1}{\varepsilon} \rceil + 1 \) s.t. 

Each \( a \delta \) and \( b \delta \) are in the same subinterval.

So \( |(a \delta) - (b \delta)| \leq \varepsilon \).
We get

\[ ad = \lfloor a \rfloor d + \{ a \} d^2 \quad \text{①} \]
\[ bd = \lfloor b \rfloor d + \{ b \} d^2 \quad \text{②} \]

Subtract ① from ② \( \Rightarrow \)

\[ (b - a) d = (\lfloor b \rfloor d - \lfloor a \rfloor d) + (\{ b \} d^2 - \{ a \} d^2) \]

\( \therefore \)

\( q = b - a, \quad p = \lfloor b \rfloor d - \lfloor a \rfloor d \)

\[ 1 \leq p \leq q \leq \left\lfloor \frac{1}{\varepsilon} \right\rfloor. \]

Thm [Roth]: For any \( \varepsilon > 0 \) there are only finitely many rationals \( \frac{p}{q} \) that satisfy

\[ \left| d - \frac{p}{q} \right| \leq \frac{1}{q^2 + \varepsilon}. \]