Facts about random regular graphs

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Defn: A graph is $d$-regular if every vertex has degree $d$.

Figure: The Petersen graph is 3-regular with 10 vertices.

Regular graphs provide simple models to study sparse graphs (bounded average degree).

Useful for both theoretical and practical purposes (physical or social networks, etc.).
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If $nd$ is even then there are a finite number of such graphs $G$.

So we may consider one chosen uniformly at random.

However, ...
- It is difficult to compute probabilities with this model.
- How many $d$-regular graphs on $n$ vertices are there?
- If $G$ is chosen uniformly at random, what is $\mathbb{P}[G \text{ is triangle free}]$?
  What about $\mathbb{P}[G \text{ is bipartite}]$?

Must define a different model of random regular graphs; one amenable to combinatorial analysis.
The configuration model

Introduced by [Bender-Canfield, 78] and also [Bollobás, 80].

- Start with $n$ vertices: $1, 2, \ldots, n$.
- Each vertex receives $d$ labelled half edges.
- Pair these $nd$ half edges at random and glue each pair into a full edge.

Figure: A 3-regular graph on 4 vertices.
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Each simple graph occurs in $(d!)^n$ different pairings. Hence,

$$\Pr [G_{n,d} \text{ is simple}] = \frac{(d!)^n \# \text{simple d-regular grasps on } n \text{ vertices}}{(nd - 1)!!}.$$ 

In particular, if $G$ is simple then

$$\Pr [G_{n,d} = G \mid G_{n,d} \text{ is simple}] = \frac{1}{\# \text{simple d-regular graphs}}.$$
Theorem (Bender-Canfield, 78 and Bollobás, 80)

\[ P \left[ G_{n,d} \text{ is simple} \right] \to e^{-\frac{d^2-1}{4}} \text{ as } n \to \infty. \] More generally, if \( girth(G_{n,d}) \) is the length of the shortest cycle in \( G_{n,d} \) then

\[ P \left[ girth(G_{n,d}) > g \right] \to e^{-\sum_{k=1}^{g} \frac{(d-1)^k}{2k}}. \]
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Corollary: Let \( A_n \) be a graph property associated to \( G_{n,d} \), e.g., \( G_{n,d} \) is triangle free. If \( P \left[ A_n \right] \to 0 \) then \( P \left[ A_n \mid G_{n,d} \text{ simple} \right] \to 0 \) as well.
Independent sets in random regular graphs

**Definition**

An independent set in a graph is a subset of vertices with no edges between them.

The independence ratio of a graph $G$, denoted $\alpha(G)$, is the maximum value of $|I|/|G|$, where $I$ is an independent set in $G$. 

**Theorem (Bollobás, 81)**

For every $d \geq 3$ and $\epsilon > 0$, the probability $P[\alpha(G_n, d) \leq 2 \log d + \epsilon d]$ → 1 as $n \to \infty$.

**Corollary:**

For $d \geq 10$, $P[G_n, d \text{ is bipartite}] \to 0$ as $n \to \infty$.

Indeed, $G_n, d$ would otherwise contain an independent set of size density $1/2 > 2(\log d)/d$. 
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Corollary: For $d \geq 10$, $\mathbb{P} [G_{n,d} \text{ is bipartite}] \rightarrow 0$ as $n \rightarrow \infty$.

Indeed, $G_{n,d}$ would otherwise contain an independent set of size density $1/2 > 2(\log d)/d$. 


Proof outline of thm: Let $N_k$ be the number of independent sets of size $k$ in $G_{n,d}$.

The configuration model provides for an easy calculation:

$$
\mathbb{E} [N_k] \leq O(\sqrt{n}) \times e^{n \Phi(k/n,d)},
$$

where $\Phi(\alpha, d) = \alpha \log d - \frac{\alpha^2}{2} d$.
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The configuration model provides for an easy calculation:

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where $\Phi(\alpha, d) = \alpha \log d - \frac{\alpha^2}{2} d$.

$\Phi(\alpha, d) < 0$ if $\alpha > 2(\log d)/d$. Thus, with $\alpha = (2 \log d + \epsilon)/d$ and $k = \lceil \alpha n \rceil$ we deduce

$$\mathbb{P}\left[\alpha(G_{n,d}) > \frac{2 \log d + \epsilon}{d}\right] = \mathbb{P}[N_k \geq 1] \leq \mathbb{E}[N_k] = e^{-c_{\epsilon,d} n}$$

for some $c_{\epsilon,d} > 0$. 
Lower bound for independence ratio

Frieze and Łuczak provide an elegant, non-constructive argument for a matching lower bound.

**Theorem (Frieze-Łuczak, 92)**

For all sufficiently large values of $d$,

$$\mathbb{P} \left[ \alpha(G_{n,d}) \geq \frac{2 \log d - 2 \log \log d}{d} \right] \rightarrow 1 \quad \text{as } n \rightarrow \infty.$$ 

This would make a good topic for the short presentation.
Local structure of random regular graphs

Although $G_{n,d}$ is not bipartite, it does look bipartite \textit{locally}. 
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In fact, the local structure of $\mathcal{G}_{n,d}$ is tree-like:

Let $X_r$ be the number of vertices in $\mathcal{G}_{n,d}$ whose $r$-neighbourhood is not a tree (contains a cycle). Then

$$\mathbb{E}[X_r] = o(n) \text{ for } r \leq \frac{(1 - \epsilon) \log_{d-1} n}{2}.$$ 

That is, $1 - o(1)$ proportion of the vertices (as $n \to \infty$) have the property that their $(\kappa \log_{d-1} n)$-neighbourhood is a tree for any $\kappa < 1/2$. 
Proof:

If the \( r \)-neighbourhood of \( v \) is not a tree then it contains a cycle of length at most \( 2r \).

If a cycle \( C \) is contained in the \( r \)-neighbourhood of \( v \) then \( v \) is within distance \( x = r - \frac{|C|}{2} \) of \( C \).

The number of such \( v \) for given \( C \) is at most \( |C|d(d - 1)^{x-1} \). Therefore,

\[
X_r \leq \sum_{\text{cycles } C:|C| \leq 2r} d|C|(d - 1)^{x-1}.
\]

\[
\mathbb{E}[X_r] \leq 2rd \sum_{\ell=1}^{2r} \mathbb{E}[\#\{\ell - \text{cycles in } G_{n,d}\}] (d - 1)^{r-\frac{\ell}{2}}.
\]
Counting short cycles:

**Lemma**

Let $C_\ell$ be the number of cycles of length $\ell$ in $G_{n,d}$, where $\ell = O(\log n)$ as $n \to \infty$. Then $\mathbb{E}[C_\ell] = O((d - 1)^\ell/\ell)$ as $n \to \infty$. In fact, if $\ell$ is a fixed integer then

$$\mathbb{E}[C_\ell] \to (d - 1)^\ell/(2\ell) \text{ as } n \to \infty.$$
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It follows from the lemma that for some constant $C_d$,

$$\mathbb{E}[X_r] \leq C_d r \log r (d - 1)^{2r} = O(n^{1-\frac{\epsilon}{2}})$$

so long as $r \leq \frac{(1-\epsilon) \log_{d-1} n}{2}$. 
In conclusion

- Random regular graphs are easier to study in the configuration model.
- Graph properties that hold with probability tending to 1 in the configuration model also do so for random regular simple graphs.
- The largest independent sets in random regular graphs have size density of order $2(\log d)/d$.
- Although random regular graphs are not bipartite, they are locally tree-like due to a lack of short cycles.