18.211 Midterm Exam Solutions

1) Provide the combinatorial definition of each of the following terms.
   • The binomial coefficient \( \binom{n}{k} \): It is the number of ways to choose a \( k \)-element subset from an \( n \)-element set.
   • The Stirling number of the 2nd kind \( S(n, k) \): It is the number of ways to partition an \( n \)-element set into \( k \) non-empty subsets.
   • The \( n \)-th Fibonacci number \( F_n \): It is the number of ways to tile a \( 1 \times (n - 1) \) board using squares and dominoes (assuming that \( n \geq 1 \)).

2) How many arrangements of TALLAHASSEE are there such that the first L and first the S are adjacent in either order? For example, TALSASHALEE and TAASLASHALEE are valid arrangements but TALASSHALEE is not.

Solution: There are 2 Ls, 2 Ss, 3 As, 2 Es, 1 H and 1 T in the word. Since the number of Ls and Ss are the same, the number of arrangements where the first L and S appear together as LS equals the number of arrangements where the first L ans S appear together as SL.

We now count arrangements of the former type. There are two ways to arrange the remaining L and S after the first LS. So we have the template

\[ \wedge_1 LS \wedge_2 X \wedge_3 Y \wedge, \quad \text{where} \quad X Y = L S \text{ or } S L. \]

The remaining 7 letters can be inserted into the 4 wedges in

\[ \binom{7 + 4 - 1}{4 - 1} \]

ways.

These can then be arranged in

\[ \frac{7!}{3!2!} \]

ways.

So the total number of valid arrangements is

\[ 2 \cdot 2 \cdot \binom{10}{3} \cdot \frac{7!}{3!2!} \cdot \]

3) A restaurant has 4 identical circular tables. How many ways are there for 20 customers to sit around the tables if there are 10 people on one table, 4 on another, and 3 each on the remaining two tables?

Solution: This counts the number of permutations of \([20]\) whose cycle type is one 10-cycle, one 4-cycle and two 3-cycles. It equals

\[ \frac{20!}{2! \cdot 10 \cdot 4 \cdot 3^2}. \]
4) Give a combinatorial proof of the identity

\[ \binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2. \]

**Solution:** The LHS counts the number of ways to choose \( n \) people from a set of \( 2n \) people. On the RHS this is counted by conditioning on the number of people what are chosen from the first \( n \) set of persons. Say this number equals \( k \) for \( 0 \leq k \leq n \). Then there are \( \binom{n}{k} \) ways to choose the \( k \) persons from the first \( n \) and \( \binom{n}{n-k} = \binom{n}{k} \) ways to choose the remaining \( n - k \) persons from the last \( n \). By the product and sum rules the total number of ways to choose \( n \) people from \( 2n \) equals the RHS.