Counting arrangements using wedges

Problem 1
How many arrangements of MATHEMATICS have each consonant adjacent to a vowel?

Solution
We have 7 consonants and 4 vowels. Each vowel can have a consonant on either side, which gives us this template:

\[ \wedge_1 V \wedge_2 \wedge_3 V \wedge_4 \wedge_6 V \wedge_6 \wedge_7 V \wedge_8 . \]

This has 8 spaces for consonants but we need only fill 7, so we discard one of the spaces. There are 5 ways to do this: eliminating the 1-st or 8-th are two, the other 3 come from discarding one from an adjacent pair - note that the selection within the pair is irrelevant.

We have counted the restricted arrangements of generic consonants and generic vowels. All we need to do now is count the subarrangements of consonants and vowels: \( \frac{7!}{2!2!} \) and \( \frac{4!}{2!} \). All of these enumerations are independent of one another. We us the product rule to obtain the final answer:

\[ 5 \cdot \frac{7!}{2!2!} \cdot \frac{4!}{2!} \]
Problem 2

How many arrangements of INSTRUCTOR have exactly 2 consonants between consecutive vowels?

Solution

We have 3 vowels and 7 consonants so the restriction immediately gives us the following template of generic vowels and consonants:

\[ \wedge_1 VCCVCCV \wedge_2. \]

This leaves 3 generic consonants to be distributed between the two wedges. This can be done in 4 ways - 0, 1, 2, or 3 consonants in the first wedge justifies this count.

Now that we have counted arrangements of generic consonants and vowels, we can count subarrangements of consonants and vowels: \( \frac{7!}{2!2!} \) and 3!, respectively. All of the enumerations we have done are mutually independent. We use the product rule to get the solution:

\[ 4 \cdot \frac{7!}{2!2!} \cdot 3!. \]
Problem 3

How many $k$-element subsets of $[n]$ have no consecutive numbers? Derive both an expression and recursion.

Solution

The $k$-element subset of $[n]$ are in bijection with binary sequences of length $n$ containing $k$ ones. Avoiding consecutive numbers means that every 1 except for the last is followed by a zero, giving the following template:

$$\land_1 10 \land_2 10 \cdots \land_{k-1} 10 \land_k 1 \land_{k+1}.$$  

We must distribute the remaining $n - 2k + 1$ zeroes in the $k + 1$ wedges, which is the problem of distribution $n - k$ indistinguishable balls into $k + 1$ distinct boxes. The number of ways to do this is

$$\binom{n - 2k + 1 + (k + 1) - 1}{(k + 1) - 1} = \binom{n - k + 1}{k}.$$

To derive a recursion, let $g(n, k)$ be the number of ways to choose the $k$-element subsets. Condition on the number $n$. If it is not selected then we need only count the number of $k$-element subsets of $[n - 1]$ with no consecutive elements, which is $g(n - 1, k)$. If the number $n$ is selected then the number $n - 1$ must be excluded. This gives a $(k - 1)$-element subset of $[n - 2]$ with no consecutive elements; there are $g(n - 2, k - 1)$ of those. As these two cases are mutually exclusive we get that

$$g(n, k) = g(n - 1, k) + g(n - 2, k - 1).$$
Problem 4

What is the number of circular $k$-choices from $[n]$ in which every chosen integer is followed clockwise by at least $w$ non-chosen integers?

Solution

Equating circular $k$-choices as circularly arranged binary sequences consider the template

$$\land_1 1 \land_2 1 \cdots \land_k 1 \land_{k+1},$$

where each $\land_i$ for $i = 2, \ldots, k$ contains at least $w$ zeroes and $\land_1 \cup \land_{k+1}$ together contains at least $w$ zeroes in order to satisfy the given condition.

There are $w$ ways in which $\land_1$ can contain fewer than $i$ zeroes where $i < w$ (it can contain $0, 1, \ldots, w - 1$ zeroes). When this occurs $\land_{k+1}$ must contain at least $w - i$ zeroes. This leaves $n - k - wk$ zeroes to be freely distributed among $\land_2, \ldots, \land_{k+1}$, which can be done in

$$\binom{n-k-wk+k-1}{k-1} = \binom{n-wk-1}{k-1}$$

ways. As these cases can occur in $w$ distinct ways (depending on the value of $i$), the number of choices is $w \cdot \binom{n-wk-1}{k-1}$.

The other case is that $\land_1$ contains $w$ or more zeroes. In this case we can freely distribute the remaining $n - k - wk$ zeroes among all the $k + 1$ wedges in $\binom{n-wk}{k}$ ways.

Combining all the cases gives the final answer:

$$w \cdot \binom{n-wk-1}{k-1} + \binom{n-wk}{k}.$$